Temporal Logic with Reference Pointers

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Abstract. An extension of the propositional temporal language is introduced with a simple syntactic device, called "reference pointer" which provides for making references within a formula to "instants of reference" specified in the formula. The language with reference pointers $\mathcal{L}_{trp}$ has a great expressive power (e.g. Kamp's and Stavi's operators as well as Prior's clock variables are definable in it), especially compared to its frugal syntax, perspicuous semantics and simple deductive system. The minimal temporal logic $\mathbf{K}_{trp}$ of this language is axiomatized and strong completeness theorem is proved for it and extended to an important class of extensions of $\mathbf{K}_{trp}$. The validity in $\mathcal{L}_{trp}$ is proved undecidable.

1 Introduction

The rapidly expanding scope of applications (actual or potential) of temporal logic to theoretical computer science and artificial intelligence demands, inter alia, strengthening of the expressive power of the temporal language to make it a really appropriate tool for adequate treatment of the various temporal phenomena, while keeping a relatively simple and efficient mechanism for derivations, convenient for applications and for implementation of automated deduction systems. This demand is particularly relevant for the propositional temporal languages. Their most valuable assets are the perspicuity of the syntax and deductive apparatus on the one hand, and on the other hand their *intensional* semantic nature which allows for representation of sophisticated first- or higher-order schemata on a propositional syntactic level. These two assets are in mutual controversy, reflecting the fundamental controversy in logic: "expressiveness vs. tractability". A number of temporal languages and systems have been devised in seeking the best compromise in this controversy.

This article is intended as a further contribution in this trend. It proposes a rather simple but particularly strong extension of the propositional temporal language $\mathcal{L}_t$. As it usually happens, this enterprise was motivated from dissatisfaction with the expressiveness of $\mathcal{L}_t$. One of the major drawbacks, at least from point of view of natural language, is its lack of means to make references to

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particular points in time, somehow specified in formal context. Simply speaking, one just cannot say "now", "then", "when" in the classical temporal language. (For thorough discussion on "now" and "then" the reader is referred to e.g. [13] and [**?**]). This is sometimes particularly annoying because, as a consequence, very basic and natural features of the temporal frame cannot be expressed syntactically. To give just one example, the simple fact that "*now* will not occur again", or formally, that the flow of time is irreflexive, is beyond the expressive abilities of $\mathcal{L}_t$. Various enrichments of this language have been proposed to improve its expressiveness. Let us mention three basic types of enrichments. The first one extends the language with new particular *operators* intended to express and formalize specific temporal phenomena, e.g. Kamp's binary temporal modalities *Since* and *Until* and the more sophisticated Stavi's connectives $\mathbf{U}'(p,q)$ and $\mathbf{S}'(p,q)$ (see e.g. [4, 5] and [7]; see the latter as well for a general approach to this type of enrichments of the temporal language). The second one provides the language with new *sorts* of syntactic objects, (constants, variables etc.) having specific interpretation in the temporal framework and thus increases generally the expressive power of the language. A characteristic example are Prior's "clock-variables" ([14]; see also [3], the *nominals* in [2], and the *names* in [9]) which are bound to be true at *exactly one instant* of the flow of time. And the third one increases the expressiveness of the language by adding more rules of inference intended to depict semantic features not expressible by means of formulae. A notable example here is Gabbay's "irreflexivity rule" (see [6]).

What is proposed in this article can be atributed to each of these types of enrichments of the language. We extend the language with a specific syntactic device intended to enable making references to points in time, but the result turns out to be a significant general improvement of the expressiveness of the language. The idea in nutshell is this: we want to specify an instant in our formal temporal expression, to which we want to be able to make references further. In order to do that, when reach our *instant of reference*, we say "*now*" (i.e. fix a *point of reference*), and further, when we want to make reference to that "*now*", we put a *reference pointer* i.e. say "*then*". Schematically it looks like this: $\ldots now(\ldots\ then\ \ldots\ then\ \ldots\ then\ \ldots)\ldots$, which we formally express like this: $\ldots\downarrow(\ldots\uparrow\ldots\uparrow\ldots\uparrow\ldots)\ldots$. This construction can be iterated, e.g $\ldots\downarrow(\ldots\uparrow\ldots\downarrow(\ldots\uparrow\ldots)\ldots\uparrow\ldots)\ldots$ etc. The language can be further extended with more than one sets for reference "now-then" but even as such it is rich enough both to express Kamp's and Stavi's operators, and to simulate Prior's clock-variables. This makes it an appropriate medium for formalization of various particular systems for linear and branching time; a few basic examples are given at the end.

In the article we introduce the extended language $\mathcal{L}_{trp}$, its semantics, briefly comment on the expressiveness and prove that the satisfiability problem is undecidable. Then we give an axiomatization of the basic temporal logic $\mathbf{K}_{trp}$ for which we prove a strong completeness theorem and generalize this result to an important class of extensions of $\mathbf{K}_{trp}$. The reader is assumed to have a background in propositional temporal logics (syntax, semantics, deductive systems and completeness theorem) within the bounds of either of e.g. [1, 5, ?]. Following a referee's suggestion, a reference is also made to [16] (unfortunately not accessible to the author) where reportedly similar ideas have been discussed.

2 Syntax

The language $\mathcal{L}_t$ of the propositional temporal logic contains a countable set $P = \{p_1, p_2, \ldots\}$ of propositional variables, logical constants $\bot, \top$, connectives $\neg$ and $\wedge$, and temporal modalities **G** ("always in the future") and **H** ("always in the past"). The symbols $\vee, \rightarrow, \leftrightarrow$, **F** ("sometime in the future") and **P** ("sometime in the past") are definable in a standard way.

We extend the language $\mathcal{L}_t$ with three additional symbols:

- *universal modality* **A** ("always"), whose dual $\neg\mathbf{A}\neg$ is denoted by **E** ("sometime");
- *reference pointer* $\uparrow$ ("then"), and
- *point of reference* $\downarrow$.

The resulting language is denoted by $\mathcal{L}_{trp}$. The symbol $\uparrow$ behaves syntactically like a propositional variable, while $\downarrow$ is an unary connective which resembles a quantifier binding $\uparrow$. The recursive definition of formulae in $\mathcal{L}_t$ is extended for $\mathcal{L}_{trp}$ with the following clauses:

- $(\uparrow)$ $\uparrow$ is a formula;
- $(\mathbf{A})$ If φ is a formula then $\mathbf{A}\varphi$ is a formula;
- $(\downarrow)$ If φ is a formula then $\downarrow\varphi$ is a formula.

We need a few syntactic notions borrowed from first-order logic:

> The first occurrence of $\downarrow$ in the formula $\downarrow\varphi$ has a *scope* φ.
>
> An occurrence of $\uparrow$ in a formula φ is *bound* if it is in the scope of an occurrence of $\downarrow$; otherwise it is *free*.
>
> If φ and ψ are formulae, $\varphi(\psi/\uparrow)$ denotes the result of simultaneous substitution of all free occurrences on $\uparrow$ in φ by ψ.
>
> A formula φ is *closed* if there are no free occurrences of $\uparrow$ in φ.
>
> *Complexity* of a formula φ of $\mathcal{L}_{trp}$ is the number of logical connectives (including **A**, $\uparrow$ and $\downarrow$) in φ.
>
> *Reference depth* of a formula φ is the largest number $r(\varphi)$ of nested occurrences of $\downarrow$ in φ.

3 Semantics

We shall deal only with relational semantics for propositional temporal logic. The notions of (temporal) *frame*, *valuation* and *model* are the standard ones. Given a model $\mathcal{M} = < T, R, V >$ and a point (*instant*) $t \in T$ we have a recursive definition of truth $\mathcal{M} \models \varphi[t]$ for all formulae of $\mathcal{L}_t$, which we want to extend over the new symbols. However, we have no suitable way to define truth at an instant for non-closed formulae, since this truth would depend on the "instant of reference" which is not determined if the formula is not closed. To avoid this obstacle we choose another way to define truth at an instant, viz. by translation of the formulae of $\mathcal{L}_{trp}$ into the first-order language L_1 containing a binary predicate R and a countable set of unary predicates $\{P_1, P_2, \ldots\}$. For convenience we list the set of individual variables of L_1 as $\{x, w, y_0, y_1, y_2, \ldots\}$ since x and w will play special roles, viz. x will represent the actual point in time (the current "now"), and w will represent the instant of reference ("then"). Now we extend the translation denoted by ST in [1] as follows:

1. $ST(p_i) = P_i x$,
2. $ST(\uparrow) \;=\; x = w$,
3. $ST(\neg\varphi) = \neg ST(\varphi)$,
4. $ST(\varphi \wedge \psi) = ST(\varphi) \wedge ST(\psi)$,
5. $ST(\mathbf{G}\varphi) = \forall y(Rxy \to ST(\varphi)(y/x))$,
6. $ST(\mathbf{H}\varphi) = \forall y(Ryx \to ST(\varphi)(y/x))$,
7. $ST(\mathbf{A}\varphi) = \forall y(ST(\varphi)(y/x))$,
8. $ST(\downarrow \varphi) = ST(\varphi)(x/w)$.

In 5, 6, and 7 above y is the first variable different from x and w, which does not occur in $ST(\varphi)$; u/v means uniform substitution of u for all free occurrences of v.

Note that x and w can only have free occurrences in $ST(\varphi)$, where they are the only possibly free variables. Furthermore, φ is closed if and only if w does not occur in $ST(\varphi)$.

The model $\mathcal{M} = < T, R, V >$ can be regarded as an L_1-model where R is interpreted by R and P_i by $V(p_i), i = 0, 1, 2, \ldots$. In order to distinguish validity in $\mathcal{M}$ as an L_1-model from validity in $\mathcal{M}$ as a temporal model we shall use the symbol $|\!\vdash$ for the former case and $\models$ for the latter. Now we define for any *closed* formula φ:

$$\mathcal{M} \models \varphi[t] \;\text{ if }\; \mathcal{M} \mid\!\vdash ST(\varphi)(t/x),$$

and

$$\mathcal{M} \models \varphi \;\text{ if }\; \mathcal{M} \models \varphi[t] \;\text{ for every } t \in T, \;\text{ i.e. if }\; \mathcal{M} \mid\!\vdash \forall x ST(\varphi).$$

Finally, φ is *valid in a frame* if it is valid in every model on the frame, and φ is (*universally*) *valid* if it is valid in every temporal frame.

We only define validity for closed formulae since only they have a determined meaning. Hereafter we shall not be interested in non-closed formulae.

4 Notes on Expressiveness and Definability in $\mathcal{L}_{trp}$

The language $\mathcal{L}_{trp}$ has a great expressive power. Here we shall not give complete characterization of its expressiveness (this will be done elsewhere) but shall only present a few eloquent testimonials for it.

1. Various postulates for temporal frames, which are beyond the scope of $\mathcal{L}_t$ are readily expressed in $\mathcal{L}_{trp}$. Just two simple examples:
   - irreflexivity: $F \Vdash \forall x \neg Rxx \;\Leftrightarrow\; F \models \downarrow \mathbf{G} \neg \uparrow$,
   - antisymmetry: $F \Vdash \forall x \forall y (x < y \to \neg y < x) \Leftrightarrow F \models \downarrow \mathbf{GG} \neg \uparrow$.
2. A number of modalities not definable in $\mathcal{L}_t$ can be easily defined in $\mathcal{L}_{trp}$. A simple but important example is the *difference modality* $[\neq]$ (see e.g. [17]):

$$[\neq]\varphi = \downarrow \mathbf{A}(\neg \uparrow \to \varphi).$$

   Kamp's $\mathbf{S}(p,q)$ (*Since*) and $\mathbf{U}(p,q)$ (*Until*) and Stavi's $\mathbf{U}'(p,q)$ and $\mathbf{S}'(p,q)$ are explicitly definable here, too:

$$\mathbf{U}(p,q) = \downarrow \mathbf{F}(p \wedge \mathbf{H}(\mathbf{P} \uparrow \to q)) \text{ and } \mathbf{S}(p,q) = \downarrow \mathbf{P}(p \wedge \mathbf{G}(\mathbf{F} \uparrow \to q));$$

   $\mathbf{U}'(p,q) =$

$$\downarrow \mathbf{FH}(\mathbf{P} \uparrow \to q) \wedge \neg \mathbf{U}(\neg q \vee \neg \mathbf{U}(\top, q), q) \wedge \downarrow \mathbf{F}(\neg q \wedge p \wedge \mathbf{H}(\mathbf{P} \uparrow \wedge \mathbf{P}(\mathbf{P} \uparrow \wedge \neg q) \to p))$$

   and likewise for $\mathbf{S}'(p,q)$.
3. The idea of *clock-variables* or *names for instants* can be adequately formalized in $\mathcal{L}_{trp}$ without introducing a separate sort for variables: The formula $\downarrow \mathbf{A}(p \leftrightarrow \uparrow)$ says "p is valid only now", and accordingly, $\mathbf{E} \downarrow \mathbf{A}(p \leftrightarrow \uparrow)$ means "p is valid at exactly one instant". Thus, in the consequent of the formula $\mathbf{E} \downarrow A(p \leftrightarrow \uparrow) \to \varphi(p, \ldots)$ the variable p plays a role of a clock-variable. This fact will be essentially exploited in our axiomatic system.

As we showed above, $\mathcal{L}_{trp}$ is at least as strong as a temporal language with difference modality or clock-variables (these two are equivalent with respect to definability, shown in [9]). Therefore some results about these two languages (see [9, 17]) hold for $\mathcal{L}_{trp}$, too:

- every finite frame is described up to isomorphism in $\mathcal{L}_{trp}$.
- all universal sentences in the monadic second-order language for R and $=$ are definable in $\mathcal{L}_{trp}$.

In fact, $\mathcal{L}_{trp}$ is stronger than any of the above mentioned languages. Indeed, the formula

$$\mathbf{A}(\mathbf{F}\top \wedge \downarrow \mathbf{G} \neg \uparrow) \to \mathbf{E} \downarrow \mathbf{E}(\mathbf{FF} \uparrow \wedge \mathbf{G} \neg \uparrow)$$

says that if a frame is irreflexive and every point has a successor then it is not transitive. It is valid in every *finite* model but not in the frame $< \mathcal{N}, <>$. Therefore this condition is not definable in any of those languages, since their minimal logics enjoy the finite model property. Moreover, as one could expect about such a powerful language, the set of valid formulae in $\mathcal{L}_{trp}$ is not recursive.

Theorem 1. *The satisfiability problem in $\mathcal{L}_{trp}$ is Π^0_1-complete.*

Proof. We show that the *unbounded tiling problem* for $\mathcal{N} \times \mathcal{N}$, known to be Π^0_1-complete (see [12]), is reducible to the satisfiability problem in $\mathcal{L}_{trp}$. The idea for doing this we borrow from [18].

First we define a formula GRID which is suppose to set the grid for tiling:

$$GRID = (p \wedge q) \wedge \varphi_1 \wedge \varphi_2 \wedge \varphi_3$$

where:

$$\begin{aligned}
\varphi_1 = \mathbf{A}((&p \wedge q \to \mathbf{F}(p \wedge \neg q) \wedge \mathbf{F}(\neg p \wedge q) \wedge \mathbf{G}((p \wedge \neg q) \vee (\neg p \wedge q))) \ \wedge \\
&(p \wedge \neg q \to \mathbf{F}(p \wedge q) \wedge \mathbf{F}(\neg p \wedge \neg q) \wedge \mathbf{G}((p \wedge q) \vee (\neg p \wedge \neg q))) \ \wedge \\
&(\neg p \wedge q \to \mathbf{F}(\neg p \wedge \neg q) \wedge \mathbf{F}(p \wedge q) \wedge \mathbf{G}((\neg p \wedge \neg q) \vee (p \wedge q))) \ \wedge \\
&(\neg p \wedge \neg q \to \mathbf{F}(\neg p \wedge q) \wedge \mathbf{F}(p \wedge \neg q) \wedge \mathbf{G}((\neg p \wedge q) \vee (p \wedge \neg q)))),
\end{aligned}$$

$$\begin{aligned}
\varphi_2 = \mathbf{A} \downarrow ((&p \wedge q \to \mathbf{A}(\mathbf{F}\uparrow \to \mathbf{G}(p \wedge q \to \uparrow))) \ \wedge \\
&(p \wedge \neg q \to \mathbf{A}(\mathbf{F}\uparrow \to \mathbf{G}(p \wedge \neg q \to \uparrow))) \ \wedge \\
&(\neg p \wedge q \to \mathbf{A}(\mathbf{F}\uparrow \to \mathbf{G}(\neg p \wedge q \to \uparrow))) \ \wedge \\
&(\neg p \wedge \neg q \to \mathbf{A}(\mathbf{F}\uparrow \to \mathbf{G}(\neg p \wedge \neg q \to \uparrow)))),
\end{aligned}$$

$$\begin{aligned}
\varphi_3 = \mathbf{A} \downarrow ((&p \wedge q \to \mathbf{A}((\neg p \wedge \neg q \wedge \mathbf{FF}\uparrow) \to \mathbf{GG}(p \wedge q \to \uparrow))) \ \wedge \\
&(p \wedge \neg q \to \mathbf{A}((\neg p \wedge q \wedge \mathbf{FF}\uparrow) \to \mathbf{GG}(p \wedge \neg q \to \uparrow))) \ \wedge \\
&(\neg p \wedge q \to \mathbf{A}((p \wedge \neg q \wedge \mathbf{FF}\uparrow) \to \mathbf{GG}(\neg p \wedge q \to \uparrow))) \ \wedge \\
&(\neg p \wedge \neg q \to \mathbf{A}((p \wedge q \wedge \mathbf{FF}\uparrow) \to \mathbf{GG}(\neg p \wedge \neg q \to \uparrow)))).
\end{aligned}$$

The formula $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ says that every point of the model has exactly two successors; at one of them the valuation of p changes and the valuation of q remains the same (that would be the move "to the right"), while at the other (the move "upwards") the opposite happens. Moreover, by φ_3, the routes "right;up" and "up;right" converge. That will be enough to embed a copy of $\mathcal{N} \times \mathcal{N}$ into any model of GRID.

Now, consider a tiling problem with a set of tiles $T = \{t_1, ..., t_m\}$ and colours $C = \{c_1, ..., c_k\}$. Every tile has four sides: "up", "down", "left" and "right", each coloured in one of the colours from C. To every colour c_i we assign four propositional variables u_i ("up"), d_i ("down"), l_i ("left"), and r_i ("right"). Each tile t with sides "up", "down", "left" and "right" coloured respectrively in $c_{i_1}, c_{i_2}, c_{i_3}$, and c_{i_4}, we represent by the formula

$$\theta_t = (u_{i_1} \wedge \bigwedge_{j \neq i_1} \neg u_j) \wedge (d_{i_2} \wedge \bigwedge_{j \neq i_2} \neg d_j) \wedge (l_{i_3} \wedge \bigwedge_{j \neq i_3} \neg l_j) \wedge (r_{i_4} \wedge \bigwedge_{j \neq i_4} \neg r_j).$$

Now we define the formulae

$$COVER_T = \mathbf{A}(\bigvee_{i=1}^{m} \theta_i),$$

which says that the model is properly tiled, i.e. every point in the model is covered by exactly one tile (note that θ_i and θ_j are incompatible when $i \neq j$);

$$\begin{aligned} MATCHUP = \mathbf{A}(\bigwedge_{i=1}^{k} (u_i \to & (p \wedge q \to \mathbf{G}(p \wedge \neg q \to d_i)) \wedge \\ & (p \wedge \neg q \to \mathbf{G}(p \wedge q \to d_i)) \wedge \\ & (\neg p \wedge q \to \mathbf{G}(\neg p \wedge \neg q \to d_i)) \wedge \\ & (\neg p \wedge \neg q \to \mathbf{G}(\neg p \wedge q \to d_i))), \end{aligned}$$

which says that the colour "up" of each tile of the cover matches the colour "down" of the one above it;

$$\begin{aligned} MATCHRIGHT = \mathbf{A}(\bigwedge_{i=1}^{k} (r_i \to & (p \wedge q \to \mathbf{G}(\neg p \wedge q \to l_i)) \wedge \\ & (p \wedge \neg q \to \mathbf{G}(\neg p \wedge \neg q \to l_i)) \wedge \\ & (\neg p \wedge q \to \mathbf{G}(p \wedge q \to l_i)) \wedge \\ & (\neg p \wedge \neg q \to \mathbf{G}(p \wedge \neg q \to l_i))), \end{aligned}$$

which says that the colour "right" of each tile of the cover matches the colour "left" of the one to the right of it.

Finally, we put

$$\Phi_T = GRID \wedge COVER_T \wedge MATCHUP \wedge MATHCHRIGHT.$$

We claim that Φ_T is satisfiable if and only if $\mathcal{N} \times \mathcal{N}$ can be properly tiled by T.

Indeed, if $\mathcal{N} \times \mathcal{N}$ can be tiled by T we can define a model $\mathcal{M} =< \mathcal{N} \times \mathcal{N}, R, V >$ where

$$< m_1, n_1 > R < m_2, n_2 > \text{ iff } \; m_2 = m_1 + 1, n_2 = n_1 \; \text{ or } \; m_2 = m_1, n_2 = n_1 + 1;$$

$$V(p) = \{< 2m, n >: m, n \in \mathcal{N}\}, \;\; V(q) = \{< m, 2n >: m, n \in \mathcal{N}\},$$

$$V(u_i) = \{< m, n >: \text{the "up" colour of the tile at} < m, n > \text{is } c_i\},$$

and likewise for d_i, l_i, and r_i.

Then,

$$\mathcal{M} \models \Phi_T[< 0, 0 >].$$

Conversely, if for some model $\mathcal{M} =< W, R, V >$, $\mathcal{M} \models \Phi_T[x]$, we define a mapping $f : \mathcal{N} \times \mathcal{N} \to W$ as follows: $f(0,0) = x$; suppose that $f(m,n)$ is defined, then $f(m+1,n)$ is the unique "right"-successor of $f(m,n)$, and $f(m,n+1)$ is the unique "up"-successor of $f(m,n)$. The tiling of $\mathcal{N} \times \mathcal{N}$ is now determined by COVER: for any $< m, n >$ there is a unique θ_i which is true at $f(m,n)$; then we put the tile t_i at $< m, n >$. Due to MATCHUP and MATCHRIGHT the tiling is a proper one, and this completes the proof. □

5 Deductive Systems in $\mathcal{L}_{trp}$

Here we axiomatize the basic logic $\mathbf{K}_{trp}$ of $\mathcal{L}_{trp}$ as follows:

Axioms:

$A1)$ The axioms of the basic temporal logic $\mathbf{K}_t$, written over propositional variables. .

$A2)$ $\mathbf{S5}(\mathbf{A})$

$A3)$ $\mathbf{A}p \to (\mathbf{G}p \wedge \mathbf{H}p)$

$A4)$ $\downarrow\uparrow$

$A5)$ $\downarrow\mathbf{A}(\uparrow\leftrightarrow p) \to (q \to \mathbf{A}(p \to q))$

$A6)$ $\downarrow\mathbf{A}(\uparrow\leftrightarrow p) \to (\downarrow\psi \leftrightarrow \psi(p/\uparrow))$,

where p, q are propositional variables.

Rules:

1. **MP**:
$$\frac{\varphi, \varphi \to \psi}{\psi};$$
2. $\mathbf{NEC_A}$:
$$\frac{\varphi}{\mathbf{A}\varphi};$$
3. **CLSUB**:
$$\frac{\varphi}{clsub(\varphi)},$$
where $clsub(\varphi)$ is a result of uniform substitution of *closed* formulae for propositional variables in φ.
4. **WITNESS**:
$$\frac{\downarrow\mathbf{A}(\uparrow\leftrightarrow p) \to \varphi \text{ for every propositional variable } p}{\varphi}.$$

Note the following:

- Only closed formulae are deducible in $\mathbf{K}_{trp}$.

– In the presence of **CLSUB** the infinitary rule **WITNESS** can be replaced by a finitary version:

$$\mathbf{WITNESS}_f : \frac{\downarrow \mathbf{A}(\uparrow \leftrightarrow p) \to \varphi \text{ for some prop. variable } p \text{ not occurring in } \varphi}{\varphi}$$

– The rules

$$\mathbf{NEC_G} : \frac{\varphi}{\mathbf{G}\varphi} \ \ and \ \ \mathbf{NEC_H} : \frac{\varphi}{\mathbf{H}\varphi}$$

are derivable from $\mathbf{NEC_A}$,**MP** and A3.

Here are some important theorems of $\mathbf{K}_{trp}$:

$(t1)$ $\downarrow\varphi \leftrightarrow \varphi$ for every closed formula φ;
$(t2)$ $\downarrow\neg\varphi \leftrightarrow \neg\downarrow\varphi$;
$(t3)$ $\downarrow(\varphi \wedge \psi) \leftrightarrow (\downarrow\varphi \wedge \downarrow\psi)$;
$(t4)$ $\downarrow\mathbf{A}(\uparrow\leftrightarrow p) \to$ p.

We exemplify derivations in $\mathbf{K}_{trp}$ sketching a proof of $(t2)$:

1. $\downarrow\mathbf{A}(\uparrow\leftrightarrow p) \to (\downarrow\neg\varphi \leftrightarrow \neg\varphi(p/\uparrow))$ by A6,
2. $\downarrow\mathbf{A}(\uparrow\leftrightarrow p) \to (\downarrow\varphi \leftrightarrow \varphi(p/\uparrow))$ by A6,
3. $\downarrow\mathbf{A}(\uparrow\leftrightarrow p) \to (\neg\downarrow\varphi \leftrightarrow \neg\varphi(p/\uparrow))$ by 2 and contrapositions,
4. $\downarrow\mathbf{A}(\uparrow\leftrightarrow p) \to (\neg\downarrow\varphi \leftrightarrow \downarrow\neg\varphi)$ by 1 and 3,
5. $\neg\downarrow\varphi \leftrightarrow \downarrow\neg\varphi$ by 4 and **WITNESS**.

The other derivations are similar.

Remark 1. 1. The reason we design our axiomatic system for closed formulae is that we have defined validity only for these formulae. Besides that, however, the formulae which are not closed behave rather irregularly. For instance, the rule for equivalent replacement

$$\frac{\varphi \leftrightarrow \psi}{\theta(\varphi/p) \leftrightarrow \theta(\psi/p)},$$

applied to such formulae, does not always preserve validity in a frame. The same happens with the substitution rule $\frac{\varphi}{sub(\varphi)}$, so that special provisions must be made, like "a formula free for substitution in a formula" etc. which would lead to the typical complications of the first-order machinery.

2. The intuition behind the rule **WITNESS** (which has a number of ancestors, e.g. some versions of rules for quantifiers in first-order logic, the "irreflexivity rule" in [6], COV in [9] etc.) is the following. If a formula is not valid, then it is false at some instant t of some model $\mathcal{M}$. Then, a propositional variable p can be made a "t o'clock-variable" i.e. evaluated to be true exactly at that instant t, and then p will be a "witness" of the falsity of φ. Therefore if all clock-variables testify that φ is true at the instants in which they live, then φ must be valid.

3. Although **WITNESS** and **WITNESS**$_f$ infer the same theorems, they yield deductive systems different with respect to logical consequence which is compact in the system with **WITNESS**$_f$, but not in the system with **WITNESS**.

Amongst the basic syntactic properties of $\mathbf{K}_t$ which are transferred, mutatis mutandis, to $\mathbf{K}_{trp}$ we only mention the following

Lemma 1. *For any closed formulae α, β, φ and variable p, if*

$$\mathbf{K}_{trp} \vdash \alpha \leftrightarrow \beta$$

then

$$\mathbf{K}_{trp} \vdash \varphi(\alpha/p) \leftrightarrow \varphi(\beta/p).$$

Proof. (Sketch) Induction on the reference depth $r(\varphi)$.

If $r(\varphi) = 0$, the proof repeats the standard one for $\mathbf{K}_t$.

If $r(\varphi) = r > 0$ we assume that for all formulae with reference depth less than r the statement holds, and then do induction on the complexity of φ. The interesting case is $\varphi = \downarrow\theta$. Here we apply axiom $A6$ (with a variable q not occurring in φ, α, β) which replaces $\varphi(\alpha/p)$ by $\{\theta(\alpha/p)\}(q/\uparrow) = \{\theta(q/\uparrow)\}(\alpha/p)$ and likewise for $\varphi(\beta/p)$. The resulting formulae have reference depths lesser than r. Applying the inductive hypothesis for them, followed by application of **WITNESS**$_f$, completes the induction. □

Theorem 2 **((Soundness theorem)).**

1. *All axioms of $\mathbf{K}_{trp}$ are valid.*
2. *All rules of $\mathbf{K}_{trp}$ preserve validity in a frame, and hence preserve universal validity.*

Proof. 1) For A1 the result comes from $\mathbf{K}_t$; for A2 from **S5**; for A3 and A4 it is quite simple. As for A5, it is enough to note that
$\forall x ST(\downarrow\mathbf{A}(\uparrow\leftrightarrow p_i) \to (p_j \to \mathbf{A}(p_i \to p_j))) =$
$\forall x(\forall y(y = x \leftrightarrow P_i y) \to (P_j x \to \forall y(P_i y \to P_j y)))$ which is universally valid.
Finally, take $A6$. $ST(\psi(p/\uparrow))$ is obtained from $ST(\psi)$ by replacing all occurrences of the kind $y = w$ (which are the only possible occurrences of w in $ST(\psi)$) by $P_i y$. Due to the antecedent $\forall y(y = x \leftrightarrow P_i y)$ this is equivalent to replace all occurrences of $y = w$ by $y = x$. The result of this substitution is exactly $ST(\psi(x/w)) = ST(\downarrow\psi))$.

2) The only interesting case is the rule **WITNESS**. Suppose that for some model $< T, V >$ and instant $t \in T$, $< T, V > \not\models \varphi[t]$. Choose a variable p not occurring in φ and change the valuation V to V' as follows: $V'(p) = \{t\}$, and V' coincides with V elsewhere. Then $< T, V' > \models \downarrow\mathbf{A}(\uparrow\leftrightarrow p)[t]$ and $< T, V' > \not\models \varphi[t]$, hence $< T, V' > \not\models \downarrow\mathbf{A}(\uparrow\leftrightarrow p) \to \varphi[t]$. □

Now we set ourselves to prove completeness of $\mathbf{K}_{trp}$. Basically we follow an elaborated version of the traditional in modal and temporal logic "canonical model technique", further developed in [15, 9]. The basic steps of the proof will be scrupulously outlined in a series of lemmata, but the standard details in their proofs will be usually omitted.

First we introduce another syntactic notion (originating from the *admissible forms* in [10]; see also [9]). Let $*$ be a symbol not belonging to $\mathcal{L}_{trp}$. We define recursively *universal forms* of $*$ as follows:

1. $*$ is a universal form of $*$.
2. If $u(*)$ is a universal form of $*$ and φ is a closed formula then $\varphi \to u(*)$, $\mathbf{G}u(*)$, $\mathbf{H}u(*)$ and $\mathbf{A}u(*)$ are universal forms of $*$.

Every universal form of $*$ can be represented (up to tautological equivalence) in a uniform way:

$$u(*) = \varphi_0 \to \mathbf{L}_1(\varphi_1 \to \ldots \mathbf{L}_k(\varphi_k \to *) \ldots)$$

where $\mathbf{L}_1, \ldots, \mathbf{L}_k \in \{\mathbf{A}, \mathbf{G}, \mathbf{H}\}$ and some of $\varphi_1, \ldots, \varphi_k$ may be $\top$ if necessary.

For every universal form $u(*)$ and a formula θ we denote by $u(\theta)$ the result of substitution of θ for $*$ in $u(*)$. Obviously, if θ is a closed formula then $u(\theta)$ is a closed formula, too.

Now we introduce the rule

$$\mathbf{WITNESS}_U : \frac{u(\downarrow \mathbf{A}(\uparrow \leftrightarrow p) \to \varphi) \quad \text{for every propositional variable } p}{u(\varphi)}$$

where u is an arbitrarily fixed universal form.

Although $\mathbf{WITNESS}_U$ seems much stronger than **WITNESS**, they are in fact equivalent. The key observation for that (see [8]) is the following: given a universal form

$$u(*) = \varphi_0 \to \mathbf{L}_1(\varphi_1 \to \ldots \mathbf{L}_k(\varphi_k \to *) \ldots)$$

we define a form

$$u^{'}(*) = \neg * \to (\varphi_k \to \mathbf{L}_k^{'}(\varphi_{k-1} \to \ldots \mathbf{L}_1^{'} \neg \varphi_0) \ldots),$$

where $\mathbf{A}^{'} = \mathbf{A}, \mathbf{G}^{'} = \mathbf{H}$ and $\mathbf{H}^{'} = \mathbf{G}$. Now, for every closed formula θ, $u(\theta)$ is deductively equivalent to $u'(\theta)$ in the sense that $\mathbf{K}_{trp} + u'(\theta) \vdash u(\theta)$ and $\mathbf{K}_{trp} + u(\theta) \vdash u'(\theta)$.

Definition 1. *1. A* theory *in $\mathcal{L}_{trp}$ is a set of closed formulae of $\mathcal{L}_{trp}$, which contains all theorems of $\mathbf{K}_{trp}$ and is closed with respect to* **MP**.

*2. A W-*theory (witnessed theory) *is a theory in $\mathcal{L}_{trp}$ which is closed with respect to $\mathbf{WITNESS}_U$.*

Note that for every set of closed formulae Γ there is a minimal W-theory $\mathrm{WTh}(\Gamma)$ /a minimal theory $\mathrm{Th}(\Gamma)$/ containing Γ. Indeed, the set of all closed formulae is a W-theory. Furthermore, the intersection of every family of W-theories is a W-theory. Then $\mathrm{WTh}(\Gamma)$ is the intersection of all W-theories containing Γ. Likewise for theories.

Definition 2. *A theory /W-theory/ is consistent if it does not contain* $\perp$. *A set of closed formulae* Δ *is W-consistent if* $WTh(\Delta)$ *is consistent.*

The following property, well-known for theories, hold for W-theories, too.

Lemma 2 ((Deduction theorem for W-theories)). *If* Γ *is a* W*-theory and* φ, ψ *are closed formulae then* $\varphi \to \psi \in \Gamma$ *iff* $\psi \in WTh(\Gamma \cup \{\varphi\})$

Proof. If $\varphi \to \psi \in \Gamma$ then, by **MP**, $\psi \in \mathrm{WTh}(\Gamma \cup \{\varphi\})$. Vice versa, suppose that $\psi \in WTh(\Gamma \cup \{\varphi\})$ and consider the set

$$\Delta = \{\theta : \theta \text{ is } a \text{ closed formula and } \varphi \to \theta \in \Gamma\}.$$

We shall prove that Δ is a W-theory containing $\Gamma \cup \{\varphi\}$. The proof goes as in the standard deduction theorem, with one additional step: closedness with respect to $\mathbf{WITNESS}_U$, which follows form the fact that Γ is a W-theory, and $\varphi \to u(*)$ is a universal form whenever $u(*)$ is. $\square$

Lemma 3. *If* Γ *is a set of closed formulae in which infinitely many propositional variables have no occurrences then* $WTh(\Gamma) = Th(\Gamma)$.

Proof. It is a standard fact that

$$\mathrm{Th}(\Gamma) = \{\theta : \gamma_1 \wedge \ldots \wedge \gamma_k \to \theta \in \mathbf{K}_{trp} \text{ for some } \gamma_1, \ldots, \gamma_k \in \Gamma\}.$$

Let us show that $\mathrm{Th}(\Gamma)$ is closed with respect to $\mathbf{WITNESS}_U$. Suppose that for some universal form u, $u(\downarrow \mathbf{A}(\uparrow\leftrightarrow p) \to \varphi) \in \mathrm{Th}(\Gamma)$, i.e. $\gamma_1^i \wedge \ldots \wedge \gamma_{k_i}^i \to u(\downarrow \mathbf{A}(\uparrow\leftrightarrow p_i) \to \varphi) \in \mathbf{K}_{trp}$ for every propositional variable p_i. We can choose a propositional variable p_j which does not occur in either of u, φ, or Γ. Then, substituting in $\gamma_1^j \wedge \ldots \wedge \gamma_{k_j}^j \to u(\downarrow \mathbf{A}(\uparrow\leftrightarrow p_j) \to \varphi)$ any variable p for p_j we obtain $\gamma_1^j \wedge \ldots \wedge \gamma_{k_j}^j \to u(\downarrow \mathbf{A}(\uparrow\leftrightarrow p) \to \varphi) \in \mathbf{K}_{trp}$ for every p. Therefore $\gamma_1^j \wedge \ldots \wedge \gamma_{k_j}^j \to u(\varphi) \in \mathbf{K}_{trp}$ by $\mathbf{WITNESS}_U$, hence $u(\varphi) \in \mathrm{Th}(\Gamma)$. $\square$

As a corollary, every set of formulae which satisfies the condition of Lemma 3 is W-consistent iff it is consistent.

Definition 3. *A* W*-theory* Γ *is* maximal *if for every closed formula* φ*, either* $\varphi \in \Gamma$ *or* $\neg\varphi \in \Gamma$ *but not both.*

Every maximal W-theory is consistent and cannot be extended to another consistent W-theory. Moreover, every maximal W-theory Γ contains a "witness" $\downarrow \mathbf{A}(\uparrow\leftrightarrow p)$ for some propositional variable p ("Γ o'clock variable"). For otherwise all $\neg \downarrow \mathbf{A}(\uparrow\leftrightarrow p)$ would be in Γ, and hence, by $\mathbf{WITNESS}_U$, $\perp$ would belong to Γ.

Lemma 4 ((Lindenbaum lemma)). *Every* W*-consistent set* Γ_0 *can be extended to a maximal* W*-theory.*

Proof. First, note that $\Gamma = \mathrm{WTh}(\Gamma_0)$ is a consistent W-theory. Let $\psi_1, \psi_2, \dots$ be a list of all closed formulae of $\mathcal{L}_{trp}$ and $u_1, u_2, \dots$ be a list of all universal forms in $\mathcal{L}_{trp}$. Then we can list all combinations $\{u_i(\psi_j)\}_{i,j=1}^{\infty}$ in a sequence $\theta_1, \theta_2, \dots$.(Obviously, there will be repetitions in this sequence, but that does not matter). Now we shall define a sequence of consistent W-theories $T_0 \subseteq T_1 \subseteq \dots$ as follows: $T_0 = \Gamma$; suppose that T_n is defined and consider $\mathrm{WTh}(T_n \cup \{\theta_n\})$. If it is consistent, this is T_{n+1}. Otherwise let $\theta_n = u_i(\psi_j)$. Then $\neg u_i(\psi_j) \in T_n$ by the deduction theorem. Therefore $u_i(\downarrow\mathbf{A}(\uparrow\leftrightarrow p) \to \psi_j)$ does not belong to T_n for some propositional variable p. Then put

$$T_{n+1} = \mathrm{WTh}(T_n \cup \{\neg u_i(\downarrow\mathbf{A}(\uparrow\leftrightarrow p) \to \psi_j)\}). \text{ Finally, put } T = \bigcup_{n=0}^{\infty} T_n.$$

By virtue of the construction, T is a maximal W-theory. □

For any set of formulae Δ we define

$$\mathbf{G\Delta} = \{\varphi : \mathbf{G}\varphi \in \mathbf{\Delta}\},\ \mathbf{H\Delta} = \{\varphi : \mathbf{H}\varphi \in \mathbf{\Delta}\} \text{ and } \mathbf{A\Delta} = \{\varphi : \mathbf{A}\varphi \in \mathbf{\Delta}\}.$$

Lemma 5. *If Δ is a maximal W-theory then $\mathbf{G\Delta}, \mathbf{H\Delta}$ and $\mathbf{A\Delta}$ are W-theories.*

Proof. We shall do the proof for $\mathbf{G\Delta}$, the others are analogous.
That $\mathbf{G\Delta}$ contains all theorems of of $\mathbf{K}_{trp}$ and is closed with respect to **MP** is nothing new. $\mathbf{G\Delta}$ is also closed with respect to $\mathbf{WITNESS}_U$ since Δ is closed and $\mathbf{Gu}(*)$ is a universal form whenever $u(*)$ is. □

Lemma 6. *If Δ is a maximal W-theory and $\mathbf{F}\theta \in \mathbf{\Delta}$ / $\mathbf{P}\theta \in \mathbf{\Delta}$ / $\mathbf{E}\theta \in \mathbf{\Delta}$ respectively then there is a maximal W-theory Δ' such that $\theta \in \Delta'$ and $G\Delta \subseteq \Delta'$ / $H\Delta \subseteq \Delta'$ / $A\Delta \subseteq \Delta'$ respectively.*

Proof. By Lemma 5 $\mathbf{G\Delta}$ is a W-theory. Moreover, $\mathbf{G}\neg\theta \notin \mathbf{\Delta}$ since Δ is consistent. Therefore $\neg\theta \notin \mathbf{G\Delta}$, hence $\mathrm{WTh}(\mathbf{G\Delta} \cup \{\theta\})$ is consistent. Then, by Lemma 4 it can be extended to a maximal W-theory Δ'. The other cases are analogous. □

Definition 4. *A model $< T, R, V >$ is called* clock-model *if for every $t \in T$ there is a "t o'clock-variable" p_t such that $V(p_t) = \{t\}$.*

Lemma 7 ((Strong completeness theorem for W-consistent sets in $\mathbf{K}_{trp}$)). *Every W-consistent set Γ_0 is satisfiable in a clock-model.*

Proof. First, we extend Γ_0 to a maximal W-theory Γ. Then we define a *canonical* model $\mathcal{M} =< T, R, V >$ as follows:

- $T = \{\Delta : \Delta \text{ is a maximal } W\text{-theory and } \mathbf{A\Gamma} \subseteq \mathbf{\Delta}\}$;
- for any $\Delta_1, \Delta_2 \in T$, $R\Delta_1\Delta_2$ if $\mathbf{G\Delta}_1 \subseteq \Delta_2$;
- for any propositional variable $p, V(p) = \{\Delta \in T : p \in \Delta\}$.

It is a standard task to prove that for any $\Delta_1, \Delta_2 \in T$, $R\Delta_1\Delta_2$ iff $\mathbf{H\Delta}_2 \subseteq \Delta_1$ and that $\mathbf{A\Delta}_1 \subseteq \Delta_2$.

Now we are going to prove that Γ_0 is satisfied at the point Γ of the model $\mathcal{M}$. This will follow from the following sub-lemma:

> **Truth-lemma:** For every closed formula θ and $\Delta \in T$,
>
> $$\mathcal{M} \models \theta[\Delta] \text{ iff } \theta \in \Delta.$$
>
> *Proof of the Truth-lemma*: Induction on $r(\theta)$. when $r(\theta) = 0$, i.e. θ contains no reference pointers, the proof goes by induction on the complexity of θ and repeats, mutatis mutandis, the proof the the truth-lemma for $\mathbf{K}_t$, as the universal modality **a** is dealt with in the same way as the temporal modalities. Now, let $r(\theta) > 0$ and assume that for all closed formulae with reference depth less than $r(\theta)$ the statement holds. Then we do again an induction on the complexity of θ. The only non-standard case is $\theta =\downarrow\psi$. Let $\downarrow\mathbf{A}(\uparrow\leftrightarrow p)$ be a "witness" in Δ. Then, by axiom $A6, \downarrow\psi \leftrightarrow \psi(p/\uparrow) \in \Delta$, i.e. $\downarrow\psi \in \Delta$ iff $\psi(p/\uparrow) \in \Delta$. Since $r(\downarrow\psi) > r(\psi(p/\uparrow))$, by the inductive hypothesis $\mathcal{M} \models \psi(p/\uparrow)[\Delta]$ iff $\psi(p/\uparrow) \in \Delta$. To complete the proof of the lemma it remains to show that $\mathcal{M} \models \psi(p/\uparrow)[\Delta]$ iff $\mathcal{M} \models\downarrow\psi[\Delta]$. Again by axiom $A6$ $\mathcal{M} \models\downarrow\mathbf{A}(\uparrow\leftrightarrow p) \to (\downarrow\psi \leftrightarrow \psi(p/\uparrow))$, hence it is enough to show that $\mathcal{M} \models\downarrow\mathbf{A}(\uparrow\leftrightarrow p)[\Delta]$. $ST(\downarrow\mathbf{A}(\uparrow\leftrightarrow p)) = \forall y(y = x \leftrightarrow Py)$ where P is the unary predicate symbol corresponding to p. Thus, $\mathcal{M} \models\downarrow\mathbf{A}(\uparrow\leftrightarrow p)[\Delta]$ iff $\mathbf{M} \Vdash \forall y(y = \Delta \leftrightarrow Py)$ which means that $V(p) = \{\Delta\}$, i.e. p is a "Δ o'clock variable". Indeed, this is so: First, $p \in \Delta$ by the $\mathbf{K}_{trp}$-theorem $(t2)$ and **MP**. Now, suppose that $p \in \Delta'$ for some $\Delta' \in$T. Take any $\chi \in \Delta$. According to axiom $A4, \mathbf{A}(p \to \chi) \in \Delta$, hence $p \to \chi \in \Delta'$, so $\chi \in \Delta'$. Thus $\Delta \subseteq \Delta'$, which implies $\Delta = \Delta'$. So, $p \in \Delta'$ iff $\Delta' = \Delta$. The sub-lemma is proved, which completes the proof of Lemma 7.

□

Theorem 3 ((Strong completeness theorem for $\mathbf{K}_{trp}$)). *Every consistent set Γ of closed formulae in $\mathcal{L}_{trp}$ is satisfiable.*

Proof. With a simple trick we reduce the theorem to Lemma 7. Let ρ be a "renaming" of the propositional variables in $\mathcal{L}_{trp}$ as follows: $\rho(p_i) = p_{2i+1}, i = 1, 2, \ldots$. If φ is a formula, denote by $\rho(\varphi)$ the result of uniform substitution $\rho(p_i)$ for each p_i in φ, and then put $\rho(\Gamma) = \{\rho(\varphi) : \varphi \in \Gamma\}$. Now, $\rho(\Gamma)$ is a consistent set, since consistency is not affected by the renaming ρ. Furthermore, since the variables with even indices do not occur in formulae of $\rho(\Gamma)$, it is W-consistent by Lemma 3, hence satisfiable at some instant t of a clock-model $< T, R, V >$. Now we define a valuation V' in $< T, R >$ as follows: $V'(p) = V(\rho(p))$. The resulting model $< T, R, V' >$ (which is not necessarily a clock-model) satisfies Γ at t. □

Definition 5. $\mathcal{L}_{trp}$-logic *is every simple closed extension (extension by means of closed axioms only) of* $\mathbf{K}_{trp}$.

The strong completeness theorem is provable, mutatis mutandis, for all $\mathcal{L}_{trp}$-logics:

Theorem 4. *For each $\mathcal{L}_{trp}$-logic $\mathbf{L}$, every consistent in $\mathbf{L}$ set of closed formulae is satisfied in some $\mathbf{L}$-model.*

Of course, a valuable completeness theorem would guarantee satisfiability in a model *over an $\mathbf{L}$-frame.* Very few general results in that direction exist in modal and temporal logic, but for $\mathcal{L}_{trp}$-logics there is an important one, stated in Theorem 5 below.

For any formula θ we denote $w(\theta) = \mathbf{E}{\downarrow}\mathbf{A}(\uparrow\leftrightarrow \theta)$.
$w(\theta)$ says that θ is true at exactly one instant of the model.

Definition 6. 1. *A formula φ is* witnessed *if it is of the kind*

$$\varphi = w(q_1) \wedge \ldots \wedge w(q_k) \rightarrow \psi(q_1, \ldots, q_k),$$

where ψ is a closed formula which only contains propositional variables amongst $q_1, \ldots, q_k$. In particular, φ can be a formula without propositional variables.

2. *An $\mathcal{L}_{trp}$-logic is witnessed if it is axiomatized over $\mathbf{K}_{trp}$ by means of witnessed formulae only.*

Theorem 5. *Let $\mathbf{L}$ be a witnessed $\mathcal{L}_{trp}$-logic. Every $\mathbf{L}$-consistent set of closed formulae is satisfied in some model based on an $\mathbf{L}$-frame.*

Proof. Given a witnessed formula (which we can assume to be written over $p_1, \ldots, p_k$) $\varphi = w(p_1) \wedge \ldots \wedge w(p_k) \rightarrow \psi(p_1, \ldots, p_k)$ we define the first-order formula $\alpha(\varphi) = \forall x \forall z_1 \ldots \forall z_k \sigma(ST(\psi))$, where $z_1, \ldots, z_k$ are variables not occurring in $ST(\psi)$ and $\sigma(ST(\psi))$ is the result of uniform substitution in $ST(\psi)$ of all occurrences of atomic formulae of the kind $P_i y$ by $y = z_i$ respectively, for $i = 1, \ldots$,k. It is not difficult to verify the following: for every temporal frame F, $F \models \varphi$ iff $F \Vdash \alpha(\varphi)$. Now, let φ be an axiom of $\mathbf{L}$ and $\mathcal{M} = < T, R, V >$ be a clock-model of $\mathbf{L}$. Then $\mathcal{M}$ satisfies all variants $w(p_{i_1}) \wedge \ldots \wedge w(p_{i_k}) \rightarrow \psi(p_{i_1}, \ldots, p_{i_k})$ of φ since they are theorems of $\mathbf{L}$. This implies that $< T, R >$ $\Vdash \alpha(\varphi)$ and hence $< T, R > \models \varphi$. Thus, every clock-model of a witnessed $\mathcal{L}_{trp}$-logic $\mathbf{L}$ is based on a frame for $\mathbf{L}$. Therefore, by Lemma 7 and Theorem 3, every consistent in $\mathbf{L}$ set of closed formulae is satisfied in a model based on an $\mathbf{L}$-frame. $\square$

We conclude with a few simple examples of basic kinds of temporal logics readily axiomatized as witnessed $\mathcal{L}_{trp}$-logics.

Corollary 1. *The following extensions of $\mathbf{K}_{trp}$ are strongly complete.*

1. *The* Logic of Linear (irreflexive) Time*:* $\mathbf{LT}_{trp} = \mathbf{K}_{trp}+$
   - *(irreflexivity)* $\downarrow \mathbf{G}\neg \uparrow$,
   - *(transitivity)* $\downarrow \mathbf{A}(\mathbf{FF} \uparrow \rightarrow \mathbf{F} \uparrow)$,
   - *(linearity)* $\downarrow \mathbf{A}(\mathbf{F} \uparrow \vee \uparrow \vee \mathbf{P} \uparrow)$.

2. The Logic of Forward Branching Time*:* $\mathbf{FBT}_{trp} = \mathbf{K}_{trp} +$
  - *(common histories)* $\downarrow \mathbf{APF} \uparrow$,
  - *(linear past)* $\downarrow \mathbf{GH}(\mathbf{F} \uparrow \vee \uparrow \vee \mathbf{P} \uparrow)$,
  - *(proper branching in the future)* $\downarrow \mathbf{HF} \neg \uparrow$.

3. The Logic of Discrete Linear Time*:* $\mathbf{DLT}_{trp} = \mathbf{LT}_{trp} +$
  - *(immediate predecessor)* $\downarrow \mathbf{PGG} \neg \uparrow$,
  - *(immediate successor)* $\downarrow \mathbf{FHH} \neg \uparrow$.

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