# Situated Consequence in Elementary Situation Theory

J. M. Seligman

December 20, 1994

## 1 Situation Semantics for Situation Theory

One of the main tenets of situation semantics is that there is more to the meaning of a statement than its descriptive content:[1] In addition to knowing the descriptive content, one must also know which situation is being described in order to fully understand a statement. Even on a truth-conditional account of "meaning," the descriptive content can only be half of the story since it is not truth-value determining; the truth or falsity of the statement is determined by its *propositional* content: that the situation described supports the descriptive content. The descriptive content of a statement alone is (usually) insufficient to determine whether or not the statement is true.

In his 1952 paper "Truth" ([**?**]) Austin supposed that (what is now called) the descriptive content of a statement made using words[2] is determined by certain "descriptive conventions" relating the words to world. Presumably, Austin had in mind the kind of conventions that linguistic theories describe—those that relate "forms" to "meanings"—since it is in virtue of the meaning of the words stated that a statement has the descriptive content that it does. Exactly how words have meanings and how, in combinations, they determine descriptive content is not at issue here. It is enough to note that what is usually taken to be the descriptive content of a statement is determined by its linguistic form, so the descriptive content of a sentential statement is determined by the stated sentence.[3]

Descriptive conventions are contrasted with the "demonstrative conventions" that determine which situation a given statement is about. Little has been said about what these conventions are and it is not the purpose of this paper to do so. Nonetheless, it seems fairly clear that they have more of a pragmatic character than descriptive conventions. For example, a fairly universal demonstrative convention might be that people make true statements most of the time. This convention, if adhered to, can be enough to identify which situations are being described. Non-linguistic gestures and the mutual beliefs of speaker and hearer may also be relevant.

For current purposes the most important distinction between descriptive and demonstrative conventions is that the former determines the descriptive content of statements on the basis of linguistic information alone, whereas the latter almost always requires more. Utterances of 'The cat is on the mat' can make very different statements, depending on which situation is being talked about, but all are utterances of the same sentence.

In the process of interpretation the two kinds of convention will interact considerably. If an utterance of a sentence makes a true statement then a hearer's knowledge of which sentence is uttered will limit the

[1] The following discussion supposes that statements have a unique descriptive content. This is certainly an over-simplification but the complexities of a more elaborate theory would not undermine the line of argument developed here.

[2] Although Austin's paper, and the present one, focus on statements made using words, similar remarks could be made about other kinds of statement, such as those using maps or pictures. In fact, Austin mentions maps several times.

[3] The referents of indexicals—I, you, here, now, yesterday &c.—are also needed to determine descriptive content.

possible situations that the statement could be about: An utterance of 'The cat is on the mat', if true, must be about a situation in which the cat is on the mat. The descriptive content of a much longer statement—an accident report, for example—may even be enough to uniquely identify the described situation; but only if the statement is true. Because statements can be false, the linguistic form of a statement alone is not enough to determine which situation is being described and so not enough to determine the truth-value of the statement.

## Situated Consequence

The underdetermination of propositional content—and hence truth-value—by linguistic form raises a problem for logicians. If sentences of an interpreted language do not determine truth-values, what does it mean to say that one sentence is a logical consequence of another?

Tarski's definition of logical consequence is that the conclusion is a consequence of the premises just in case in any interpretation of the non-logical words of the language which makes the premises true also makes the conclusion true. Since truth is a property of *statements* this definition of consequence does not make sense without some explanation of how mere *sentences* can have a truth-value. There are several ways of doing this:

The naive correspondence theory of truth is that a sentence is true just in case it corresponds with the facts. Clearly no sentence corresponds with *all* the facts; a sentence "corresponds with the facts" if there is *a* fact to which it corresponds. It supposed that facts are primarily non-linguistic parts of the world; for example, the fact that the cat is on the mat is something like the situation of the cat's being on the mat. The naive question of what kind of correspondence there is between a linguistic entity like 'The cat is on the mat' and a substantial part of the world like the cat's being on the mat is answered in terms of Austin's theory of truth for statements—the sentence corresponds to the facts because 'The cat is on the mat' can be used to make a true statement about the cat's being on the mat. But such a correspondence does not legitimate the attribution of truth to sentences because it is not one-to-one. The sentence 'The cat sat on the mat' corresponds to a large number of situations containing cats on mats, since many different true statements can be made using the same sentence. Even more *false* statements can be made using the same sentence and so it *fails* to correspond to an even greater number of situations in which mats do not have cats on them. If a sentence is true just because it can be used to make one true statement then a logic based on the truth-values of sentences has little chance of capturing the principles of reasoning with statements.

Another interpretation of the naive theory is that "the facts" should not be understood either universally (*all* the facts) or existentially (*some* fact) but as a definite description of one very large fact: the world. On this interpretation, a sentence is true if it can be used to make a true statement about the world. This account fairs well with so-called eternal sentences like 'The number of planets in the solar system is nine' since any use of this sentence to make a statement (presumably) describes the same situation and that situation might as well be "the world." Sentences like 'The cat is on the mat' are either rejected (as not being the kind of sentences one would want in an ideal language) or else regarded as indexical, the described situation being determined by certain aspects of the context in which the statement is made. This is not really a solution to the problem; it only shift the burden of explanation on to the "aspects of the context." The account also relies on the ontological assumption that there is such a thing as a situation which contains all the facts—an assumption from which the semantical paradoxes can be see to arise (as explained in [**?**]).

In this paper, neither of the above proposals will be followed. Instead of trying to find a way to ascribe truth-values to sentences, the definition of logical consequence will be changed. Although truth is really a property of statements rather than sentences, logical consequence can be viewed as a relationship between sentences which holds in virtue of the statements they can be used to make about situations: $\psi$ is a consequence of $\varphi$ if and only if, if a true statement can be made using $\varphi$ then a true statement *about the same situation* can also be made using $\psi$. This notion of logical consequence is called "situated consequence" because it is concerned with the logical relationships between statements describing the same situation.

The axioms and rules of classical reasoning are sound for situated consequence just as they are sound for the standard Tarksian consequence—except, perhaps, if negation and quantification are regarded as partial but in that case easy modifications can be made without losing the classical spirit of the logic (see Section 3). In fact, for many languages there is no difference between situated consequence and any other definition of logical consequence. Differences emerge only when studying the kind of language in which a *theory of situations* is expressed.

## Elementary Situation Theory

Unlike many other language users, situation theorists need to refer to situations directly; they need to have terms which denote situations and predicates which express properties of situations; they need quantifiers ranging over situations, in order to state generalizations about situations, and, most importantly, they need to make statements about situations being described by sentences (or being of situation-types or supporting infons).

Situation theorists also want to talk about entities more abstract than situations (such as infons, propositions, properties and types) and state generalizations about them but such generalizations would involve higher-order quantification and so they will not be addressed in this paper. Here, we will be concerned only with elementary (i.e., first-order) situation theory.

According to Austin's theory of truth, every statement is about a situation—the one it describes. Usually, logical connections between statments are only concerned with different descriptions of this situation; if '$\varphi \to \psi$' describes a situation also described by $\varphi$ then that same situation is described by $\psi$— so the rule of Modus Ponens holds in *any* situation. But the statements of situation theory are not just *about* situations; they have a descriptive content concerned with situations themselves and the relationships that hold between them. The logical connections between sentences of situation theory reflect these relationships in a way that sentences of other languages do not. For example, the statement that $\varphi$ describes $s$ is surely about some situation, but it need not be about $s$. If it is not, there is little one can directly infer from the statement since any inference must be about the same situation. Only when the described situation is $s$ itself can it be concluded that $\varphi$. It is the main goal of this paper to provide a systematic axiomatization of these kind of inferences.

First, the language of elementary situation theory must be defined more precisely. The aim is to stay as close as possible to the language of first-order logic whilst incorporating the linguistic devices needed for situation theory. The language of propositional logic over some set of sentence letters is extended by introducing situation-terms and formulae of the form "$\varphi$' describes $s$', where $\varphi$ is a formula and $s$ is a situation term. First-order quantifiers (binding situation-variables) are also introduced.

Already there are are more consequences than those usually axiomatized by classical logic. For example, although the entailment between 'The cat is on the mat' and 'Some cat is on the mat' is a theorem of classical logic, the entailment between "The cat is on the mat' describes situation $s$' and "Some cat is on the mat' describes situation $s$' is not; neither is the entailment between 'The cat is on the mat' and 'Some situation is described by 'The cat is one the mat". These consequences could be axiomatized as a *theory* in a classical language which was able to encode its own syntax. It would be a rather ugly theory but there would be no need to supplement the underlying logic. The logical novelties only arise when the special nature of situation-terms is considered.

Situation-terms, like sentences, are descriptions of situations; the only difference being that a situation-term describes a *unique* situations. Consequently, it is perfectly sensible to allow situation-terms to be combined by propositional operators with other situation-terms or sentences. For example, if $s$ and $t$ are both names for situations then they can be used to make statements: the statement that $s$ is true just in case it is about $s$ and the statement that $s$ and $t$ is true just in case $s$ and $t$ both name the described situation.

Just as situation-terms can be used to make statements of a special kind, so sentences can be used to name situations. For example, 'The cat is on the mat' can be thought of as a name for a situation in which the cat is on the mat. Of course, there may be very many situations in which the cat is on the mat so occurrences of sentences as names for situations are taken to have existential force: If $\varphi$ occurs as a name in a context in which a certain property $P$ is being predicated of the situation named by $\varphi$—as it does in $P(\varphi)$—then this is understood as meaning that there is some situation described by $\varphi$ which has the property $P$.

Thus, from a syntactic point of view there is no difference between situation-terms and sentences; where situation-terms can occur as arguments so can sentences and where sentences can occur in logical compounds so can situation-terms. The key difference is a semantic one: The former describe exactly one situation whereas the latter may describe many situations or none at all. This difference becomes of great importance in determining (situated) consequence.

When situation-terms occur in the premises of an argument the argument can be viewed as occurring "in" the situation named by the term and, in constructing proofs, it will be seen to be necessary to conduct different parts of the proof "in" different situations. Situation-terms will be used to keep track of which situation a given part of the proof is in. For example, to show that if $s$ is described by $\varphi \wedge \psi$ then $s$ is described by $\varphi$ it is sufficient to hypothetically "move" into $s$ and prove that if $\varphi \wedge \psi$ then $\varphi$. This can be done by adding $s$ as an extra (hypothetical) premise.

The full calculus for "situated" proofs will be given in Section 3. In the intermediate section, the analogy with spatial language, already hinted at, will be developed further.

## 2 Situated Reasoning: The Spatial Analogy

The fantasy of supposing that the situation described by a statement is the one the stater is *in* is a useful one. It is the first step in making an analogy between places and situations and between the spatial-indexicality of many statements and the way in which the meaning of any statement is dependent on the situation it describes.

The attraction of the spatial analogy is that it provides a way of motivating the rules of inference for expressions of elementary situation theory. We can characterize the logical properties of '$\varphi$ describes $s$', '$s$', '$s_1$ is the same situation as $s_2$', '$P(s)$', and '$P(\varphi)$' by studying their spatial analogues: '$\varphi$ in $l$' or 'In $l$, $\varphi$' (which will be regarded as equivalent), 'This is $l$', '$l_1$ is the same situation as $l_2$', '$P(l)$' and '$P(\varphi)$' (where $l$, $l_1$ and $l_2$ are place-name). The precise way in which these expressions are used will emerge as they are addressed later in this section.

The disadvantage of the analogy is the same as the disadvantage of all analogies: It can be over-used. We must avoid being tempted into finding situational analogues for every aspect of the use of spatial language. To aid resistance to such temptations, our use of the spatial analogy will be confined to this section of the paper.

Situated consequence is analogous to "spatial consequence": A (spatially-indexical) sentence $\psi$ is a *spatial consequence* of a (spatially-indexical) sentence $\varphi$ if and only if wherever the sentence $\varphi$ is true,[4] $\psi$ is also

[4]In this section, it will be assumed—contra Section 1—that the truth-value of a sentential statement is determined by the sentence it is a statement of and the place in which it is made. This is done in order that described situations and place-indices are not confused. The disanalogous aspect of the relationship between situated and spatial consequence are discussed at the end of the section.

true. The contribution of each of the spatial expressions listed above will be assessed by studying intuitively valid arguments which use them and formulating natural deduction rules to characterize those arguments. Particular attention will be given to arguments which either "introduce" a spatial expression in the conclusion which does not occur in the premises or "eliminate" an expression which does occur in the premises by drawing a conclusion which does not contain the expression.

For the primary spatial word 'in' the following arguments are examples of introduction and elimination:

(1) The sun is shining; this is Bloomington, so the sun is shining in Bloomington.

(2) In Tokyo, people drive on the left; this is Tokyo, so people drive on the left.

They suggest the following introduction and elimination rules for 'in':

$$\dfrac{\varphi \qquad \text{This is } l}{\text{In } l,\ \varphi}\,\textit{In}\text{-I} \qquad\qquad \dfrac{\text{In } l,\ \varphi \qquad \text{This is } l}{\varphi}\,\textit{In}\text{-E}$$

These rules should be taken as part of a natural-deduction calculus for classical logic (such as that found in [?]) so that the logical connectives behave in the usual way. A natural question is Are these enough? The answer, unfortunately, is no! The rules for 'in' are sound—no invalid arguments can be made using them—but there are intuitively valid arguments which cannot be represented. The most important arguments are those that depend on the reasoner imagining, as a hypothetical premise, that they are somewhere else. For example:

(3) Alcohol is forbidden in Abu Dabi; Sake contains alcohol; so Sake is forbidden in Abu Dabi.

This argument can be justified using the introduction and elimination rules for 'in' if, in addition, the hypothetical premise 'This is Abu Dabi' is allowed for the duration of the argument:

$$\cfrac{[\text{This is Abu Dabi}] \qquad \cfrac{\cfrac{\text{In Abu Dabi, alcohol is forbidden} \qquad [\text{This is Abu Dabi}]}{\text{Alcohol is forbidden}}\,\textit{In}\text{-E} \qquad \text{Sake contains alcohol}}{\text{Sake is forbidden}}\,\text{C.S.}}{\text{In Abu Dabi, sake is forbidden}}\,\textit{In}\text{-I}$$

(The rule C.S. is an abbreviation for a sub-proof which could be constructed from a "common sense" understanding of the logical form of the example sentences.) At the last step of the proof, the hypothetical premise is *discharged*—removed from the set of genuine premises—and this is indicated by enclosing any occurrences of the premise in brackets. Whilst this strategy works for (3), it can lead to mistakes. For example, the invalid argument

(4) In Islamabad, it only rains during monsoon; it's raining; so it's monsoon in Islamabad

appears to have a similar proof:

$$\cfrac{[\text{This is Islamabad}] \qquad \cfrac{\cfrac{\text{In Islamabad, it only rains during monsoon} \qquad [\text{This is Islamabad}]}{\text{It only rains during monsoon}}\,\textit{In}\text{-E} \qquad \text{It's raining}}{\text{It's monsoon}}\,\text{C.S.}}{\text{In Islamabad, it's monsoon}}\,\textit{In}\text{-I}$$

It is easy to see what is going wrong: The premise ‘It’s raining’ is spatially indexical—said here, one may infer that it’s raining here but not that it’s raining in Islamabad. On hypothetical journeys one should avoid burdening oneself with premises which are true at home but false abroad. This kind of mistake can be avoided if care is taken in specifying the circumstances under which a hypothetical premise of the form ‘This is $l$’ can be discharged. It turns out that a sufficient condition is that the “context” of the proof—the set of premises and conclusion after the discharge—does not contain any spatially-indexical sentences. This condition is met in the proof of argument (3) since the context consists of only ‘Alcohol is forbidden in Abu Dabi’, ‘Sake contains alcohol’ and ‘Sake is forbidden in Abu Dabi’, none of which are spatially-indexical. The condition is *not* met by the proof of argument (4) since that contains the spatially-indexical sentence ‘It’s raining’.

We are not done yet. There is another way of using hypothetical premises to obtain valid arguments which cannot be proved, even using the discharge rule described above. Consider the argument:

(5) It’s raining; wherever it’s raining, it’s wet, so it’s wet.

Suppose that ‘wherever’ is a universal quantifier over locations, so that the second premise has the logical form ‘For all locations $l$, if it’s raining in $l$ then it’s wet in $l$’. In order to use this premise, a reasoner must find a location at which to instantiate it. Of course, the reasoner would like to instatiate it at her current location, where it’s raining. To do so, she must first give a name to her current location—‘$X$’, say. This can be done by hypothetically assuming ‘This is $X$’. Then she can infer ‘It’s raining in $X$’ (by ‘in’-introduction); instatiate the second premise to ‘If it’s raining in $X$ then it’s wet in $X$’; deduce ‘It’s wet in $X$’ and conclude ‘It’s wet’ (by ‘in’-elimination). Finally, she should be able to discharge the hypothetical premise ‘This is $X$’:

$$\dfrac{[\text{This is } X] \qquad \dfrac{\dfrac{\text{It's raining} \quad [\text{This is } X]}{\text{In } X\text{, it's raining}}\,\mathit{In}\text{-I} \qquad \dfrac{\text{Wherever it's raining, it's wet}}{\text{If it's raining in } X \text{ then it's wet in } X}\,\forall\text{-E}}{\text{In } X\text{, it's wet}}\,\to\text{-E}}{\text{It's wet}}\,\mathit{In}\text{-E}$$

Such a discharge is not licenced by the condition discussed above since the context contains the spatially-indexical sentences ‘It’s raining’ and ‘It’s wet’. The discharge of ‘This is $X$’ is only safe because it uses an invented name, ‘$X$’. More precisely, there is another sufficient condition on discharges of premises of the form ‘This is $l$’: that $l$ is a place-name which does not occur in any sentence of the context.

The discharge in the proof of (5) is licenced because its context consists of the sentences ‘It’s raining’, ‘Whenever it’s raining, it’s wet’ and ‘It’s wet’, none of which contain an occurrence of ‘$X$’. The discharge of ‘This is Islamabad’ in the non-proof of (4) is not licenced because its context contains the sentences ‘In Islamabad, it only rains in the monsoon’ and ‘It’s monsoon in Islamabad’, both of which contain occurrences of ‘Islamabad’.

The introduction and elimination rules for ‘in’ together with rules for the usual classical connectives and the two discharge rules mentioned above are jointly sufficient to capture all spatial consequences involving spatial expressions of the form ‘In $l$, $\varphi$’ and ‘This is $l$’. There is also a bonus: Spatial-identity expressions of the form ‘$l_1$ is $l_2$’ can be defined as ‘In $l_1$, this is $l_2$’[5] so that the argument

(6) Cows are sacred in Banares; Banares is Varanasi, so cows are sacred in Varanasi.

[5] Equivalently, ‘In $l_2$, this is $l_1$’. This indicates a simplification of the semantics of spatial indexicals: ‘this’ is given a narrow reading.

can be given the following proof:

$$\dfrac{\dfrac{\text{Cows are sacred in Banares} \quad [\text{This is Banares}]}{\text{Cows are sacred}}\,\textit{In}\text{-E} \qquad \dfrac{\text{Banares is Varanasi} \quad [\text{This is Banares}]}{\text{This is Varanasi}}\,\textit{In}\text{-E}}{\text{Cows are sacred in Varanasi}}\,\textit{In}\text{-I}$$

Note that 'Banares is Varanasi' is defined to be 'In Banares, this is Varanasi' and that the hypothetical premise 'This is Banares' is dischargable because the context contains no spatially-indexical sentences.

The attribution of properties to places by expressions of the form '$P(l)$' has not been mentioned explicitly but many of the logical properties of such expressions are captured by the usual rules of classical logic; in particular, the use of spatial quantifiers (such as 'wherever' and 'somewhere') to express generalizations concerning places can be captured by the same rules as those for individual quantifiers.

Special to the spatial case is the attribution of *relational* properties to places. Being on the other side of the world is a relation property, so expressions such as 'Tokyo is on the other side of the world' are spatially indexical: they relate the subject, Tokyo, to the indexically-determined current location of the stater. The logical properties of expressions using relational predicates is governed by the rules given above. For example, the following examples are proved by instances of the introduction and elimination rules for 'in':

(7) This is Bloomington; Edinburgh is far away, so in Bloomington, Edinburgh is far away.

(8) In Edinburgh, London is to the south; this is Edinburgh, so London is to the South.

The attribution of relational properties to places can be extended by using expressions of the form '$P(\varphi)$' to mean that somewhere $P$-related to the current location, $\varphi$. For example, 'It's raining to the south' will be taken to mean that somewhere to the south of the current location, it's raining. The following are examples of the introduction and elimination of such expressions:

(9) One can bath in hot-spring water in Hakone; Hakone is within driving distance, so within driving distance, one can bath in hot-spring water.

(10) It's raining on the eastern horizon; wherever it's raining, it's wet, so it's wet on the eastern horizon.

Introduction and elimination rules are extracted from these examples:

$$\dfrac{\text{In } l, \varphi \qquad P(l)}{P(\varphi)}\,\textit{Rel}\text{-I} \qquad\qquad \dfrac{P(\varphi) \qquad \begin{matrix}[P(X), \quad \text{In } X, \varphi] \\ \vdots \\ \psi\end{matrix}}{\psi}\,\textit{Rel}\text{-E}$$

The elimination rule needs some explanation: It is similar to the elimination rule for '$\exists$', stating that, if one has a proof of $\psi$ from the premises of the argument together with the hypothetical premises '$P(X)$' and 'In $X$, $\varphi$' (indicated in the above by vertical dots) then one can discharge the hypothetical premises to get a proof of $\psi$ from the premises of the argument alone. Also like $\exists$-elimination, there is a condition on the choice of the place-parameter '$X$': it must not occur in any sentence in the context—in this case $\psi$ together with the non-hypothetical premises. For an example, consider the following proof of (10):[6]

[6] A proof of (9) can be obtained simply by instantiating the *Rel*-I rule.

$$\dfrac{\text{It's raining on the eastern horizon} \qquad \dfrac{\dfrac{\dfrac{\text{Wherever it's raining, it's wet}}{\text{If it's raining in } X \text{ then it's wet in } X}\,\text{C.S.} \qquad [\text{It's raining in } X]}{\text{It's wet in } X}\,{\to}\text{-E} \qquad [X \text{ is on the eastern horizon}]}{\text{It's wet on the eastern horizon}}\,\textit{Rel}\text{-I}}{\text{It's wet on the eastern horizon}}\,\textit{Rel}\text{-E}$$

This completes our logical analysis of spatial expressions. Of course, there are many logical properties of these expressions which we have said nothing about; we are only concerned to motivate patterns of inference which carry over, by analogy, to situation theory. The spatial analogy has several disanalogous aspects which it is only proper for us to record here. Firstly, the spatial preposition 'in' has many properties that are not appropriate for the relation of description. We can reason from the premises that Tokyo is in Japan and that people drive on the left in Japan, that people drive on the left in Tokyo but this kind of reasoning is not used in connection with situations and descriptions.

More seriously, the spatial analogy is suggestive of a view in which the situation described by a statement is determined in a similar way to the way in which the place a spatially-indexical statement is about is determined, i.e., indexically. This is certainly *not* the view of the author.

Finally, there is a much more serious disanalogy concerning the relationship between situated consequence and "spatial consequence" and arising because of our use of non-spatially-indexical sentences, such as those of the form 'In $l$, $\varphi$'. There is no situation-theoretic analogue to non-spatial-indexicality since *all* statements describe some situation. The problem is that non-spatially-indexical sentences can be used to make statements with the same truth-values anywhere; if an utterance of 'It's raining in London' in Tokyo is true then so is any (simultaneous) utterance of the same sentence in Bloomington, Edinburgh or anywhere else.

In the next section, the relationship of description will be formalized as if it were truth-value determining in a way analogous to the way the spatial preposition 'in' is. In other words, it will be assumed that if $\varphi$ describes the situation named by $s$ then any statement made using the sentence '$\varphi$ describes $s$' is true. This assumption is clearly false; the sentence only describes a very special kind of situation—one that relates the words of $\varphi$ to the situation named by $s$. The need to make such a false assumption marks the boundary of the methods adopted in this paper. A more accurate logic of description requires a better understanding of the relationship between utterances and the situations they describe.[7]

# 3 A Sequent Calculus

Given a countable set of *variables*, a countable set of *parameters*[8], a *signature* $\Sigma$ consisting of relation-symbols and function-symbols (of various finite arities) and a *distinguished binary relation-symbol* '$\delta$' (for 'describes'), we define *terms* $T(\Sigma)$ in the usual (inductive) way: Variables and parameters are terms and if $s_1, \ldots, s_n$ are terms and $f$ is an $n$-ary function-symbol then '$f(s_1, \ldots, s_n)$' is a term. All terms are to be understood as denoting situations.[9]

[7] An approach using "information channels" ([?], [?], [?]) is currently being pursued by the author.

[8] The parameters are the formal parameters used in proof-theory and should not be confused with the (real?) parameters of situation-theory.

[9] Other sorts could be introduced if needed.

*Well-formed formulae* (wffs) $L(\Sigma)$ are also defined inductively: All terms are atomic wffs[10] and if $\varphi_1, \ldots, \varphi_n$ are wffs and $R$ is an $n$-ary relation-symbol then '$R(\varphi_1, \ldots, \varphi_n)$' is an atomic wff;[11] if $\varphi$ and $\psi$ are wffs then so are '$(\neg\varphi)$', '$(\varphi \wedge \psi)$', '$(\varphi \vee \psi)$' and '$(\varphi \to \psi)$' and if $a$ is a parameter, $x$ is a variable and $\varphi^x_a$—meaning $\varphi$ with every occurrence of $x$ replaced by an occurrence of $a$—is a wff then so are '$(\forall x\varphi)$' and '$(\exists x\varphi)$'.

Standard conventions for the omission of parentheses will be used without notice. Also, the standard notions of subformula and of "free" and "bound" occurrences of variables will be used. For a term to be *closed* is for it to contain no free variables. For a wff to be a *sentence* is for it to contain no free variables.

An important (inductively-defined) class of wffs are the $\delta$-wffs: an atomic wff with relation-symbol '$\delta$' is a $\delta$-wff and any logical compound (using '$\neg$', '$\wedge$', '$\vee$', '$\to$', '$\forall x$', or '$\exists x$') of $\delta$-wffs is a $\delta$-wff. The importance of $\delta$-wffs is that they have a constant truth-value: If they are true of one situation then they cannot be false of another situation. A wff is *$\delta$-free* if it contains no occurrences of the symbol '$\delta$'. A wff is *parametric* if it contains a parameter; otherwise, it is *non-parametric.*

An example: Suppose 'The cat is on the mat' and 'There's a cat on the mat' are 0-ary relation-symbols (i.e., sentence letters). The expression

$$\text{`}\forall x(\delta(x, \text{The cat is on the mat}) \to \delta(x, \text{There's a cat on the mat}))\text{'}$$

is a wff which means that any situation described by 'The cat is on the mat' is also described by 'There's a cat on the mat'; it contains no free variables so it is also a sentence; it contains no parameters so it is non-parametric but it contains two occurrences of '$\delta$' so it is not $\delta$-free; moreover, it is a logical compound of atomic $\delta$-wffs, so it is a $\delta$-wff.

There follows the Gentzen-style sequent calculus $LK_\delta$. Throughout, as above, '$\varphi$' and '$\psi$' range over sentences; '$s$' and '$t$' range over closed terms; '$a$' ranges over parameters and '$\Gamma$' and '$\Delta$' range over sets of sentences. Standard abbreviations using ',' for set union on either side of '$\vdash$' will be used.

A *sequent* is a pair $\langle\Gamma, \Delta\rangle$ of sets of sentences. A *proof* (in $LK_\delta$) is a finite tree labelled by sequents such that if a node labelled by $\langle\Gamma, \Delta\rangle$ has children labelled by $\langle\Gamma_1, \Delta_1\rangle, \ldots, \langle\Gamma_n, \Delta_n\rangle$ then

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \ldots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

is an instance of one of the following *rules*:

### Axioms

$$\frac{}{\varphi, \Gamma \vdash \Delta, \varphi}\,\mathrm{A}$$

If a situation is described by $\varphi$ then, no matter what other sentences $\Gamma$ describe it, it is still described by $\varphi$. Note that all sequents of the form '$s, \Gamma \vdash \Delta, s$' are axioms since all terms are wffs, so the axioms state that the same situation is being described on both sides of '$\vdash$'.

[10] Expressions which can be used to describe situations are wffs; situation-terms are used to describe situations, so situation-terms are wffs. Since all terms are situation-terms, all terms are wffs.

[11] In particular, if $s_1, \ldots, s_n$ are terms and $R$ is an $n$-ary relation-symbol then '$R(s_1, \ldots, s_n)$' is an atomic wff. When one of the $\varphi_i$ is not a term, it should be understood existentially; '$R(\ldots, \varphi_i, \ldots)$' means that there is a situation described by $\varphi_i$ which has the relational property $\lambda x.R(\ldots, x, \ldots)$. This is analogous to the treatment of relational properties in Section 2

**Occurrence Rules**

The "occurrence" rules are needed because terms and wffs can occur in unusual places: Terms can occur in any position that a wff can occupy and wffs can occur in any position that terms can occupy. The three rules in this group are highly interactive but the first is distinguished in that it involves only terms:

$$\frac{s, \Gamma \vdash \Delta}{\Gamma', \Gamma \vdash \Delta, \Delta'}\text{T } (\Gamma, \Delta\ \delta\text{-wffs})$$

T (for 'term-occurrence') rule governs the use of terms in wff-position. In particular, it concerns the use of terms as hypothetical premises in a proof by stating that to prove an argument, it is legitimate to hypothetically assume an additional premise—that the situation being described is named by a given term. The T rule's side-condition is analogous to the first condition on hypothetical premises of the form 'This is $l$' discussed in Section 2. The context $\Gamma, \Delta$ must only contain $\delta$-wffs and any non-$\delta$-wffs $\Gamma', \Delta'$ can be disposed of. The $\delta$-wffs have constant truth-values and, in this respect, are analogous to non-spatially-indexical sentences.

The "wff-occurrence" rules govern the use of wffs in term-position, as arguments to relations:

$$\frac{\delta(\varphi, a), \psi[a], \Gamma \vdash \Delta}{\psi[\varphi], \Gamma \vdash \Delta}\text{LW } (a \text{ new}, \psi \text{ atomic}) \qquad \frac{\Gamma \vdash \Delta, \delta(\varphi, s) \quad \Gamma \vdash \Delta, \psi[s]}{\Gamma \vdash \Delta, \psi[\varphi]}\text{RW } (\psi \text{ atomic})$$

The notation '$\psi[\varphi]$' is a non-standard generalization of the standard notation for dealing with occurrences of terms in wffs; it stands for a wff in which a particular "top-level" occurrence of $\varphi$ is "marked." When '$\psi[\varphi_1]$' and '$\psi[\varphi_2]$' are used together it is understood that '$\psi[\varphi_2]$' is obtained by replacing the marked occurrence of $\varphi_1$ in '$\psi[\varphi_1]$' by an occurrence of $\varphi_2$ (or vice versa). A *top-level* occurrence of $\varphi$ in $\psi$ is one which is not (properly) inside a wff which occurs as an argument of an atomic subformula of $\psi$. For example, if '$\psi[\varphi_1]$' is

'$\varphi_1 \wedge R(\varphi_1, R(\varphi_1, s))$'

then '$\psi[\varphi_2]$' is either '$\varphi_2 \wedge R(\varphi_1, R(\varphi_1, s))$' or '$\varphi_1 \wedge R(\varphi_2, R(\varphi_1, s))$' (depending on which occurrence of $\varphi_1$ is marked) but not '$\varphi_1 \wedge R(\varphi_1, R(\varphi_2, s))$' since the third occurrence of $\varphi_1$ in '$\varphi_1 \wedge R(\varphi_1, R(\varphi_1, s))$' is not a top-level occurrence—it is inside '$R(\varphi_1, s)$' which is an argument of the atomic subformula '$R(\varphi_1, R(\varphi_1, s))$'.

The wff-occurrence rules are slightly difficult to understand at first but they are easy to apply in practice. In both rules, $\psi$ is restricted to being atomic, so '$\psi[s]$' must be either $s$ or of the form '$R(\ldots, s, \ldots)$' and so '$\psi[\varphi]$' must be either $\varphi$ or '$R(\ldots, \varphi, \ldots)$'. There are really only two central cases:

In the first case, LW (for 'wff on the left') states that if $a$ is a new parameter[12] and $\delta(\varphi, a), a, \Gamma \vdash \Delta$ then $\varphi, \Gamma \vdash \Delta$. In other words, reading backwards, in proving an argument with premise $\varphi$ it is legitimate to assume that $\varphi$ describes $a$, where $a$ is a new name for the situation being described. Similarly, RW (for 'wff on the right') states that if $\Gamma \vdash \Delta, \delta(\varphi, s)$ and $\Gamma \vdash \Delta, s$ then $\Gamma \vdash \Delta, \varphi$; in other words, an argument with conclusion $\varphi$ is justified by a proof that $\varphi$ describes $s$ and a proof that the described situation is $s$.

Notice that from this case of the LW rule, the rule

[12] '$a$ new' means that $a$ does not occur below in the line. The term $a$ is restricted to being a parameter for technical reasons: A proof containing parameters is best regarded as a function taking closed terms to proofs, so a parameter is really a bound variable over closed terms. Parameters are used in this way in the classical quantifier rules.

$$\frac{a, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}(a \text{ new})$$

is derivable. This rule enforces a condition analogous to the second condition on hypothetical spatial premises: It allows the reasoner to "name" the situation being described with a new parameter.

In the second case, LW states that if $a$ is a new parameter and $\delta(\varphi, a), R(a), \Gamma \vdash \Delta$ then $R(\varphi), \Gamma \vdash \Delta$. In other words, to use '$R(\varphi)$' as a premise, hypothetically assume '$\delta(\varphi, a)$' and '$R(a)$' for new $a$. This is analogous to the *Rel*-elimination rule for spatially indexical expressions occurring as arguments to a relational spatial predicate. The *Rel*-introduction rule has its analogue in RW, which, in this case, states that if $s$ is a term and $\Gamma \vdash \Delta, \delta(\varphi, s)$ and $\Gamma \vdash \Delta, R(s)$ then $\Gamma \vdash \Delta, R(\varphi)$; in other words, '$R(\varphi)$' can be proved from proofs that $\varphi$ describes the situation $s$ and that $R(s)$.

**Logical Rules for the Classical Connectives**

The rules for the logical symbols of conjunction, disjunction, implication, negation, universal and existential quantification are all standard. Together with the axioms, they form a complete calculus for elementary classical logic.

$$\frac{\varphi, \psi, \Gamma \vdash \Delta}{\varphi \wedge \psi, \Gamma \vdash \Delta}\text{L}\wedge \qquad \frac{\Gamma \vdash \Delta, \varphi \quad \Gamma \vdash \Delta, \psi}{\Gamma \vdash \Delta, \varphi \wedge \psi}\text{R}\wedge \qquad \frac{\varphi, \Gamma \vdash \Delta \quad \psi, \Gamma \vdash \Delta}{\varphi \vee \psi, \Gamma \vdash \Delta}\text{L}\vee \qquad \frac{\Gamma \vdash \Delta, \varphi, \psi}{\Gamma \vdash \Delta, \varphi \vee \psi}\text{R}\vee$$

$$\frac{\Gamma \vdash \Delta, \varphi \quad \psi, \Gamma \vdash \Delta}{\varphi \rightarrow \psi, \Gamma \vdash \Delta}\text{L}\rightarrow \qquad \frac{\varphi, \Gamma \vdash \Delta, \psi}{\Gamma \vdash \Delta, \varphi \rightarrow \psi}\text{R}\rightarrow \qquad \frac{\Gamma \vdash \Delta, \varphi}{\neg\varphi, \Gamma \vdash \Delta}\text{L}\neg \qquad \frac{\varphi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg\varphi}\text{R}\neg$$

$$\frac{\varphi^x_t, \Gamma \vdash \Delta}{\forall x.\varphi, \Gamma \vdash \Delta}\text{L}\forall \qquad \frac{\Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta, \forall x.\varphi^a_x}\text{R}\forall\ (a \text{ new}) \qquad \frac{\varphi, \Gamma \vdash \Delta}{\exists x.\varphi^a_x, \Gamma \vdash \Delta}\text{L}\exists\ (a \text{ new}) \qquad \frac{\Gamma \vdash \Delta, \varphi^x_t}{\Gamma \vdash \Delta, \exists x.\varphi}\text{R}\exists$$

**Logical Rules for Description**

The rules for the $\delta$ relation are analogous to the introduction and elimination rules for 'in':

$$\frac{\Gamma \vdash \Delta, s \quad \varphi, \Gamma \vdash \Delta}{\delta(\varphi, s), \Gamma \vdash \Delta}\text{L}\delta \qquad \frac{\Gamma \vdash \Delta, s \quad \Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta, \delta(\varphi, s)}\text{R}\delta$$

A proof (in $LK_\delta$) with root labelled by the sequent $\langle\Gamma, \Delta\rangle$ is said to be a proof of $\Gamma \vdash \Delta$ (in $LK_\delta$). The calculus $LK_\delta$ defines a relation $\vdash$ between sets of sentences: $\Gamma \vdash \Delta$ (in $LK_\delta$) iff there is a proof of $\Gamma \vdash \Delta$ (in $LK_\delta$). $\vdash$ is intended to be (a generalization of) the relation of situated consequence: $\Gamma \vdash \Delta$ if and only if situations described by every sentence in $\Gamma$ are also described by at least one sentence in $\Delta$.

In the remainder of this section a number of important properties of $\vdash$ are stated and discussed.

First, it is shown that the logic is almost *analytic*, in the proof-theoretic sense,—whether or not a sequent has a proof is determined by whether or not its parts have proofs. To make this condition more precise, a *quasi-subformula* of a wff $\varphi$ is defined to be either a subformula of $\varphi$ or a wff of the form '$\delta(\psi, s)$' in which $\psi$ is a subformula of $\varphi$.

**Quasi-analyticity Theorem:** The only wffs occurring in proofs of $\Gamma \vdash \Delta$ are quasi-subformulae of sentences in $\Gamma, \Delta$.

PROOF: Inspection reveals that in any application of a $LK_\delta$ rule, the wffs above the line are all quasi-subformulae of wffs below the line, so the theorem holds for any proof by induction on its tree-structure. □

**Cut Theorem:** If $\Gamma \vdash \Delta, \varphi$ and $\varphi, \Gamma' \vdash \Delta$ then $\Gamma', \Gamma \vdash \Delta, \Delta'$.

PROOF: The details of this proof—an extension of Gentzen's *Haupstatz* for the system $LK$ of classical first-order logic—are far too messy to go into here. The basic strategy is to show that the rule

$$\frac{\Gamma \vdash \Delta, \varphi \qquad \varphi, \Gamma' \vdash \Delta}{\Gamma', \Gamma \vdash \Delta, \Delta'}\text{Cut}$$

is admissible by showing that, if it is added to the calculus, then it can be eliminated from any proof which uses it. Unfortunately, it is difficult to apply this strategy directly, since the restrictions on the T rule render ineffective the usual strategies for Cut-elimination (see [?]). Instead, Cut-elimination is shown for a slightly different system $LK_\delta^-$ all of whose rules are derivable in $LK_\delta$ and such that all the rules of $LK_\delta$+Cut are derivable in $LK_\delta^-$+Cut. In other words, all sequents provable in $LK_\delta^-$ are provable in $LK_\delta$ and all sequents provable in $LK_\delta$+Cut are provable in $LK_\delta^-$+Cut, so if it can be shown, by Cut-elimination, that all sequents provable in $LK_\delta^-$+Cut are provable in $LK_\delta^-$ then all sequents provable in $LK_\delta$+Cut are provable in $LK_\delta$ and we are done. In fact, Cut-elimination for $LK_\delta^-$+Cut is no easy matter and it is more convenient to add some extra Cut-like rules and show that all Cuts in the larger system can be simultaneously eliminated. More details of this proof are given in [?]. □

**Monotonicity Theorem:** If $\Gamma \vdash \Delta$ then $\Gamma', \Gamma \vdash \Delta, \Delta'$.

PROOF: By induction on the height of proofs. □

Reflexivity on singletons (i.e., that $\varphi \vdash \varphi$) and Cut are generally regarded as minimal conditions that a consequence relation must satisfy; analyticity is a sign of good health and monotonicity is related to the existence of a certain kind of semantics—

Note that the following are all theorems of $LK_\delta$:

1. $\forall x(\delta(\varphi \wedge \psi, x) \leftrightarrow \delta(\varphi, x) \wedge \delta(\psi, x))$
2. $\forall x(\delta(\varphi \vee \psi, x) \leftrightarrow \delta(\varphi, x) \vee \delta(\psi, x))$
3. $\forall x(\delta(\varphi \rightarrow \psi, x) \leftrightarrow \delta(\varphi, x) \rightarrow \delta(\psi, x))$
4. $\forall x(\delta(\neg\varphi, x) \leftrightarrow \neg\delta(\varphi, x))$
5. $\forall x(\delta(\forall y\varphi, x) \leftrightarrow \forall y\delta(\varphi, x))$
6. $\forall x(\delta(\exists y\varphi, x) \leftrightarrow \exists y\delta(\varphi, x))$
7. $\forall x(\delta(\delta(\varphi, s), x) \leftrightarrow \delta(\varphi, s))$
8. $\forall x(\delta(\psi[\varphi], x) \leftrightarrow \exists y(\delta(\varphi, y) \wedge \delta(\psi[y], x)))$
9. $\delta(s, s)$
10. $\delta(s_1, s_2) \rightarrow (\delta(\varphi, s_1) \leftrightarrow \delta(\varphi, s_2))$

(where '$\varphi_1 \leftrightarrow \varphi_2$' abbreviates '$\varphi_1 \rightarrow \varphi_2 \wedge \varphi_2 \rightarrow \varphi_1$'.) As an example of a proof in $LK_\delta$ one arrow of 8. will be proved:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{A}
\UnaryInfC{$a, \delta(\psi[\varphi], a) \vdash \exists y(\delta(\varphi, y) \wedge \delta(\psi[y], a)), a$}
\AxiomC{}
\RightLabel{A}
\UnaryInfC{$a, \delta(\varphi, b), \psi[b] \vdash \delta(\varphi, b)$}
\AxiomC{}
\RightLabel{A}
\UnaryInfC{$a, \delta(\varphi, b), \psi[b] \vdash a$}
\AxiomC{}
\RightLabel{A}
\UnaryInfC{$a, \delta(\varphi, b), \psi[b] \vdash \psi[b]$}
\RightLabel{R$\delta$}
\BinaryInfC{$a, \delta(\varphi, b), \psi[b] \vdash \delta(\psi[b], a)$}
\RightLabel{R$\wedge$}
\BinaryInfC{$a, \delta(\varphi, b), \psi[b] \vdash (\delta(\varphi, b) \wedge \delta(\psi[b], a))$}
\RightLabel{R$\exists$}
\UnaryInfC{$a, \delta(\varphi, b), \psi[b] \vdash \exists y(\delta(\varphi, y) \wedge \delta(\psi[y], a))$}
\RightLabel{LW}
\UnaryInfC{$a, \psi[\varphi] \vdash \exists y(\delta(\varphi, y) \wedge \delta(\psi[y], a))$}
\RightLabel{L$\delta$}
\BinaryInfC{$a, \delta(\psi[\varphi], a) \vdash \exists y(\delta(\varphi, y) \wedge \delta(\psi[y], a))$}
\RightLabel{T}
\UnaryInfC{$\delta(\psi[\varphi], a) \vdash \exists y(\delta(\varphi, y) \wedge \delta(\psi[y], a))$}
\RightLabel{R$\rightarrow$}
\UnaryInfC{$\vdash \delta(\psi[\varphi], a) \rightarrow \exists y(\delta(\varphi, y) \wedge \delta(\psi[y], a))$}
\RightLabel{R$\forall$}
\UnaryInfC{$\vdash \forall x(\delta(\psi[\varphi], x) \rightarrow \exists y(\delta(\varphi, y) \wedge \delta(\psi[y], x)))$}
\end{prooftree}
$$

The ten theorems suggest a way of modelling situations by the sets of sentences that describe them: If $\delta$ is interpreted as set-membership[13] then the ten theorems can be interpreted as stating first-order conditions on a set $M$ of sets of sentences of $L(\Sigma)$. If $M$ satisfies all these conditions it is called a *set-structure* (for $L(\Sigma)$). The following theorem shows that every consistent set of sentences can be thought of as describing a single situation.

**Model Existence Theorem:** If $\Gamma$ is a consistent set of sentences (i.e., it is not the case that $\Gamma \vdash$) then there is a set-structure $M$ and a set $m \in M$ such that $\Gamma \subseteq m$.

PROOF: The existence of such a structure can be shown by adapting the Lindenbaum construction for first-order languages to this setting. Details are given in [**?**]. □

# 4 Some Logical Observations

The languages $L(\Sigma)$ contain some interesting sub-languages whose logical properties can be captured by imposing various restrictions on $LK_\delta$.

### Propositional Logics

Given a signature $\Sigma$ let $L^0(\Sigma)$ be the language defined in the same way as $L(\Sigma)$ but without quantifiers.

If $\Sigma$ is a signature only containing a countable set of sentence-letters (i.e., 0-ary relation-symbols) then the logic[14] of $L^0(\Sigma)$ is axiomatized by the classical axioms and (non-quantifier-) rules. In other words, $LK_\delta$ is a conservative extension of classical propositional logic.

If $\Sigma$ is a signature only containing a countable set of sentence-letters and a countable set of constants (i.e., 0-ary function-symbols) then the logic of non-parametric $L^0(\Sigma)$ is axiomatized by the classical axioms and (non-quantifier-) rules together with the $\delta$-rules, the T rule and the instances of the wff-occurrence rules schematized by

[13] Note that the first argument to $\delta$ is taken to refer to itself under this interpretation; for example, $\delta(s, s)$ states that '$s$' is an element of $s$.

[14] The logic of a sub-language $L$ of $L(\Sigma)$ is the relation $\vdash$ restricted to sets of wffs of $L$.

$$\frac{a, \delta(\varphi, a), \Gamma \vdash \Delta, \delta(\psi, a)}{\varphi, \Gamma \vdash \Delta, \psi} (a \text{ new})$$

in which either the $\varphi$ or the $\psi$ may be missing.

It is easily seen that the full calculus is conservative over these fragments since the rules of the fragments are the only ones that can be applied to get a proof of a sequent in the sub-language. Each of the meta-theorems for $LK_\delta$ discussed in the previous section hold for both subcalculi.

### Predicate Logic

The (standard) first-order language (with identity) generated by a signature $\Sigma$ is a sub-language of $L(\Sigma)$ if '$s = t$' is defined as '$\delta(s, t)$'. $LK_\delta$ is a conservative extension of the classical consequence relation on this sub-language since the only applicable rules are either classical axioms or rules, or rules of the propositional logic (above). A simple inductive argument on the height of proofs shows that any rule-applications of the latter kind can be eliminated.

Conversely, there is a translation of wffs of $L(\Sigma)$ by wffs of the first-order language (with identity) generated by the signature $\Sigma^{+1}$ which has the same symbols as $\Sigma$ but with each symbol having an arity one greater than the arity it has in $\Sigma$. In fact, there is such a translation $\{s\}$ for each term $s$:

$$\{s\}f(x_1, \ldots, x_n) \mapsto s = f(x_1, \ldots, x_n)$$

$$\{s\}R(x_1, \ldots, x_n) \mapsto R(s, x_1, \ldots, x_n)$$

$$\{s\}\delta(x_1, x_2) \mapsto x_1 = x_2$$

$$\{s\}F(\varphi_1, \ldots, \varphi_n) \mapsto \exists x_1 \ldots x_n(\{x_1\}\varphi_1 \wedge \ldots \{x_n\}\varphi_n \wedge F(x_1 \ldots x_n))$$

$$\{s\} \text{ logical-compound}(\psi_1, \ldots, \psi_n) \mapsto \text{logical-compound}(\{s\}\psi_1, \ldots, \{s\}\psi_n)$$

The translations establish a logical equivalence between the two languages: $\Gamma \vdash \Delta$ in $LK_\delta$ iff $\{s\}\Gamma \vdash \{s\}\Delta$ in first-order logic.

### Modal Logic

If $\Sigma$ is a signature containing a countable set of sentence-letters and a single unary relation symbol '$\Diamond$' then the logic of non-parametric $L^0(\Sigma)$ is axiomatized by the classical axioms and (non-quantifier-) rules together with the rule

$$\frac{\varphi \vdash \Delta}{\Diamond\varphi, \Gamma' \vdash \Delta', \Diamond\Delta}$$

where $\Diamond\Delta$ is the set of '$\Diamond\psi$' for each $\psi$ in $\Delta$. The modal operator $\Box$ can be defined by $\Box\varphi =$'$\neg\Diamond\neg\varphi$'. Note that a special case of this rule is

$$\frac{\vdash \phi}{\vdash \Box\phi}$$

(a.k.a. Necessitation) and that all wffs of the form '$(\Box(\phi \to \psi) \wedge \Box\phi) \to \Box\psi$' are theorems; in other words,

$LK_\delta$ is a conservative extension of the minimal normal propositional modal logic $K$.

This result can be generalized: If $\Sigma$ is any signature without function-symbols then the logic of non-parametric $L^0(\Sigma)$ is the minimal normal modal logic in which each $n$-ary relation-symbol in $\Sigma$ is an independent $n$-ary existential modal operator. Adding constants to the signature but restricting to the $\delta$-free wffs results in the languages of nominal modal logic (see [?] and [?]).

In the converse direction, there is a translation of the propositional logic of $LK_\delta$ into nominal S5: '$\delta(\phi, s)$' is translated as '$\Box(s \to \phi)$'.

### Feature Logic

If $\Sigma$ is a signature only containing a countable set of unary relation-symbols and a countable set of constants, then the wffs of the $\delta$-free sub-language of $L^0(\Sigma)$ can be thought of as describing feature-structures ([?],[?],[?]). The logic of this sub-language is axiomatized by the corresponding modal-style rules (above) but it is not quite feature logic; to axiomatize feature logic it is necessary to strengthen the modal-style rule to

$$\frac{\Gamma \vdash \Delta}{\mathrm{f}(\Gamma), \Gamma' \vdash \Delta', \mathrm{f}(\Delta)}$$

where $\mathrm{f}(\Gamma)$ is the set of wffs '$\mathrm{f}(\varphi)$' for each $\varphi$ in $\Gamma$ and 'f' is a unary relation-symbol—a *feature*. This rule ensures that a feature cannot have two incompatible values: $\mathrm{f}(\varphi), \mathrm{f}(\neg\varphi) \vdash$ is provable. It is common to think of constants as representing distinct, atomic values in a feature-structure. This restriction can be captured by adding two new axioms: For all distinct constants $c_1$ and $c_2$ and for all features f,

$$\frac{}{c_1, c_2 \vdash} \qquad\qquad \frac{}{c_1, \mathrm{f}(\varphi) \vdash}$$

Also common is the convention of writing '$\mathrm{f}_1(\ldots \mathrm{f}_n(\varphi))$' as '$\mathrm{f}_1 \ldots \mathrm{f}_n(\varphi)$'. The sequence '$\mathrm{f}_1 \ldots \mathrm{f}_n$' represents a finite path in the feature-structure. The language can be extended (as in [?]) to includes path-equations of the form '$\mathrm{f}_1 \ldots \mathrm{f}_n = \mathrm{g}_1 \ldots \mathrm{g}_m$'. These can be defined in $L(\Sigma)$ as '$\exists x(\mathrm{f}_1 \ldots \mathrm{f}_n(x) \wedge \mathrm{g}_1 \ldots \mathrm{g}_m(x))$'. If such wffs are added to $L^0(\Sigma)$ their logic can be captured by the appropriate instances of the $\exists$ and $\wedge$ rules, namely:

$$\frac{\mathrm{f}_1 \ldots \mathrm{f}_n(a), \mathrm{g}_1 \ldots \mathrm{g}_m(a), \Gamma \vdash \Delta}{\mathrm{f}_1 \ldots \mathrm{f}_n = \mathrm{g}_1 \ldots \mathrm{g}_m, \Gamma \vdash \Delta}(a \text{ new}) \qquad\qquad \frac{\Gamma \vdash \Delta, \mathrm{f}_1 \ldots \mathrm{f}_n(t) \qquad \Gamma \vdash \Delta, \mathrm{g}_1 \ldots \mathrm{g}_m(t)}{\Gamma \vdash \Delta, \mathrm{f}_1 \ldots \mathrm{f}_n = \mathrm{g}_1 \ldots \mathrm{g}_m}$$

Such an approach has been taken in [?]. Values of more complex types, such as sets or lists, can be described by adding function-symbols to the language together with axioms expressing the appropriate recursion equations.[15] Extensions of this kind are discussed in [?].

The languages $L^0(\Sigma)$ and even $L(\Sigma)$ can be used to talk about feature structures: '$\delta(\varphi, s)$' means that $\varphi$ describes the feature structure $s$ and, in general, $\delta$-wffs express constraints on feature structures. For example, the wff

$$\forall x(\delta(\mathrm{f}, c) \to \exists y \delta(\mathrm{g}, x))$$

states that every feature-structure whose f-value is $c$ is itself the value of a g-feature. The use of $\delta$-wffs adds considerable expressive power to these languages.

[15] Of course, the inductive nature of such types cannot be captured without moving to a higher-order logic.

### Partiality: Negation and Quantification

Throughout this paper it has been assumed that negation is to be understood classically. This assumption does not make much sense either in situation theory or in the spatial analogy. The easiest way to address this inadequacy is to remove '$\neg$' from the language, double-up each symbol in the signature, so there is a positive and negative version of each, and *define* a negation on logical compounds using the De Morgan equivalences. Another option is to keep the negation '$\neg$' and add a second, strong negation '$\sim$'. Sequent calculus formulations of logics containing two such negations have been given in [**?**].

Slightly more problematic is the issue of quantification. Reverting to the spatial analogy for a moment, it is clear that quantifiers can have two different interpretations:

(11) In Tokyo, every head-of-state bowed; they saluted the dead god.

(12) In London, every umbrella was up; it had been raining for days.

Each of the above have two interpretations depending on the scope of the quantifiers but, in each case, one is more natural than the other. In (11) it seems more natural to understand 'every head-of-state' as having wide scope, meaning 'every head-of-state in the world', whereas in (12) it seems more natural to understand 'every umbrella' as having narrow scope, meaning 'every umbrella in London'. Similarly, one can conceive of two different kinds of quantification in the situation-theoretic setting (as explained in Chapter 9 of [**?**]).

Throughout this paper it has been assumed that quantifiers always take wide scope. To capture the logic of narrow scope quantifiers, add a new unary relation-symbol $D$ (for 'domain') to the language and change the quantifier-rules to the following:

$$\frac{\varphi^x_t, \Gamma \vdash \Delta \qquad \Gamma \vdash \Delta, D(t)}{\forall x \varphi, \Gamma \vdash \Delta}\text{L}\forall \qquad \frac{D(a), \Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta, \forall x \varphi^a_x}\text{R}\forall\ (a \text{ new}) \qquad \frac{\varphi, D(a), \Gamma \vdash \Delta}{\exists x \varphi^a_x, \Gamma \vdash \Delta}\text{L}\exists\ (a \text{ new}) \qquad \frac{\Gamma \vdash \Delta, D(t), \varphi^x_t}{\Gamma \vdash \Delta, \exists x \varphi}\text{R}\exists$$

### Higher-order Situation Theory

This paper has been concerned with the logic of elementary descriptions of situations. No account has been given of the various higher-level entities talked about by situation-theorists—infons, properties, relations, types and propositions. A full theory of situations should state how propositions can be true or false, how objects can stand in relations in situations and many other such higher-level matters. Some of these topics are addressed in [**?**].

## Acknowledgements

The work summarised in this paper was begun in Edinburgh whilst I was a member of the Human Communication Research Centre, the Centre for Cognitive Science and the Department of Artificial Intelligence. I am grateful to these institutions for their support as well as to the DYANA project (ESPRIT Basic Research Action 1195) on which I was employed. In Edinburgh, conversations with Patrick Blackburn and Mike Reape provided interesting connections with their work on modal logic and feature logic, respectively. Thanks also to Mark Hepple and Guy Barry whose *Derivation Drawer* LaTeXmacros were invaluable. The paper itself was written at Indiana University, Bloomington. I thank all the members of the situation theory group at IU for their suggestions, especially regarding the spatial analogy. My visit to IU is supported by an SERC

Postdoctoral Fellowship. Finally, some of the technical details were sorted out whilst visiting ICOT in Tokyo. I am very grateful to members of ICOT for inviting me and making my stay so enjoyable.

# References

[Aus50] J. L. Austin. Truth. *Proceedings of the Aristotelian Society*, xxiv, 1950.

[Bar89] J. Barwise. *The Situation in Logic.* CSLI, Stanford, 1989.

[Bar91] J. Barwise. Information links in domain theory. Technical report, IU Logic Group, Bloomington, 1991.

[Bar92] J. Barwise. Constraints, channels and the flow of information. This volume.

[BE87] J. Barwise and J. Etchemendy. *The Liar: An Essay in Truth and Circularity.* Basil Blackwell, Oxford, 1987.

[Bla90] P. Blackburn. *Nominal Tense Logic and other Sorted Intensional Frameworks.* PhD thesis, University of Edinburgh, 1990.

[GG91] G. Gargov and V. Goranko. Modal logic with names. In *Colloquium on Modal Logic*, 1991.

[Joh88] M. Johnson. *Attribute-Value Logic and the Theory of Grammar.* CSLI, Stanford, 1988.

[Lan89] T. Langholm. Algorithms for partial logic. Technical Report COSMOS Report 9, Department of Mathematics, Oslo, 1989.

[Mos91] M. A. Moshier. On completeness theorems for feature logics. 1991.

[Pra65] D. Prawitz. *Natural Deduction.* Almqvist and Wiksell, Stockholm, 1965.

[Rea91] M. Reape. An introduction to the semantics of unification-based grammar formalisms. Technical Report DYANA Deliverable R3.2.A, Centre for Cognitive Science, Edinburgh, 1991.

[RK86] W. C. Rounds and R. T. Kasper. A complete logical calculus for record structures representing linguistic information. In *Proceedings of the 15th Annual IEEE Symposium on Logic in Computer Science*, Cambridge, MA, 1986.

[Sel91a] J. M. Seligman. A cut-free sequent calculus for elementary situated reasoning. Technical Report HCRC-RP 22, HCRC, Edinburgh, 1991.

[Sel91b] J. M. Seligman. Physical situations and information flow. In *Situation Theory and its Applications, II*, 1991.

[Sel92a] J. M. Seligman. Note on cut-elimination for $LK_\delta$. 1992.

[Sel92b] J. M. Seligman. Proofs with austinian propositions. 1992.