A Cut-free Sequent Calculus for Elementary Situated Reasoning *

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Abstract

A first-order language is interpreted in the following way: terms are regarded as referring to *situations* and the truth of formulae is relativized to a situation. The language is then extended to include formulae of the form $t : \phi$ (where t is a term and ϕ is a formula) meaning that ϕ is true in the situation referred to by t. Gentzen's sequent calculus for classical first-order logic is extended with rules which capture this interpretation. Variants of the calculus and extensions of the language are discussed and the *Cut* rule is shown to be eliminable from some of the proposed calculi.

Situation theory has been concerned with a range of issues centring around the partiality, context dependency and intensional structure of information. In formalizing situation theory one must focus on a specific aspect of the whole package — there is too much uncertainty and equivocation about the connections between the various parts.

A dominant approach in recent years has been to focus on building models of various situation theoretic objects, usually in the universe of non-well-founded sets or in some other category of structured objects (Barwise 1987, Aczel 1990, Fernando 1989). The aim of this research — as I understand it — is to specify the mutual connections between identity criteria for the different kinds of entity about which situation theorists talk.

Here we adopt a different approach, focusing instead on the logical (or, more precisely, the proof-theoretic) consequences of reasoning in a world of situations

*Thanks to Patrick Blackburn and Mike Reape for our on-going dialogues concerning languages with 'free' occurrences of terms and for their significant work in that area. Thanks also to Mark Hepple and Guy Barry whose *Derivation Drawer* LaTeXmacros were invaluable. This research was supported in by the Human Communication Research Centre, Edinburgh, and by DYANA (ESPRIT Basic Research Action 1195). It is due to be presented at the third conference on Situation Theory and its Applications, Japan, November 1991.

where information is supported *locally* rather than being absolute (in the classical mould) or cumulative (as the intuitionists prefer it). That said, we make a number of simplifying assumptions in this paper. Firstly, we make no attempt to capture the intensional or higher-order aspects of situation theory. By this, we mean that information in both its *infon*ic and propositional guises will be studied extensionally — roughly by the set of situations which make it true — and that there will be no quantification or abstraction over informational entities. Secondly, we do not try to account for the partiality of information in any more than a trivial way.

Since higher-order quantification, abstraction and intensionality have all been main concerns for the "modellers", this choice probably needs some justification. All we can offer at this stage is a methodological hope. From a logical point of view, these aspects of the problem have been extensively studied in other settings. There are a range of techniques available from Aczel (1980), Bealer (1982), Feferman (1984) and others. Our primary interest is in answering the question: what makes situation theory any different? The answer must lie — or so we believe — in the central dependence of situation theory on the notion of a *situation*. We hope that if the elementary logical issues concerning the characterization of reasoning *in* and *about* situations can be resolved then familiar techniques will be available for handling intensionality and higher-order quantification.

1 Principles of Situated Reasoning

In developing a "situated logic" we are motivated by two ideas as to why having situations makes a difference. Firstly, we aim to capture those principles of reasoning which can be seen to be sound for a "situated" reasoner. We suppose that such a being is *in* a particular situation and is reasoning on the basis of the information locally available. Whether the situation that the reasoner inhabits is thought of as an epistemic state or a "part of the world", or even some perspectival hybrid, is largely irrelevant to our current purposes. All that matters is that there is a relationship of "supporting" (denoted by '$\models$') between situations and items of information. Secondly, we wish to describe ways of reasoning with information *about* situations. The primary example of such information is that conveyed by propositions of the form '$s \models \sigma$' where s refers to a situation and σ expresses some information. Our initial goal is therefore to describe the logical properties of $\models$.

In keeping with the tradition of the Proof Theorists we think of the meaning of the supports relation as being given by its introduction and elimination rules within a calculus of *natural deduction* (Prawitz 1965). The introduction rule can be found by asking on what premises the situated reasoner can rationally conclude that $s \models \sigma$. Similarly, the elimination rule is found by asking what she can rationally conclude from the information that $s \models \sigma$.

We start with the elimination rule. Putting ourselves in the position of a situated reasoner (which should not be too difficult if the larger claims of situation theory are correct), we ask what we can conclude from the information that $s \models \sigma$. The natural answer — that σ holds — depends on the additional premise that the situation we currently inhabit is s. This line of thought motivates the following rule of $\models$-*elimination*:

$$\frac{s \models \sigma \qquad s}{\sigma}\models\mathrm{E}$$

The only unorthodox aspect of this rule is the occurrence of the term s as a premise. From an *un*-situated point of view it is difficult to see what could possibly be meant by an inference which had a *term* as one of its premises. For the situated reasoner the meaning is apparent: just as other information may or may not hold in the current situation, the information *that* s holds just in case the current situation *is* s.

The study of logics with terms occurring as formulae is of current interest in the fields of modal and tense logic (Blackburn 1990, Gargov & Goranko 1991) and feature logic (Reape 1991). In the modal case there is little change in the standard possible-world semantics of these languages — terms are regarded as formulae which are true at exactly one world — but there is an appreciable increase in expressive power. Nevertheless, little work has been done on the proof theory of such languages since most authors have been content with an axiomatic presentation; these being good enough for their meta-logical purposes. We hope to show that the interpretation of such languages within situation theory suggests natural and computationally efficient principles of reasoning leading to a correct and complete calculus.

It is easy to formulate an introduction rule for $\models$. One can conclude that $s \models \sigma$ from the premises that σ holds in the current situation and that the current situation is s. The rule of $\models$-*introduction*,

$$\frac{s \qquad \sigma}{s \models \sigma}\models\mathrm{I}$$

leaps to mind. However, though sound, this rule will not give us a complete calculus. There is something *more* contained in the information that $s \models \sigma$. In Situation Theory, expressions of the form $s \models \sigma$ are taken to express *propositions*. Propositions, in contrast to *infons*, have a truth value. Their truth is not situation dependent in the way that the supporting of infons is. When a situated reasoner knows that $s \models \sigma$ she knows that $s \models \sigma$ is true independently of her current situation.

It may be possible to conclude $s \models \sigma$ on the basis of *hypothetical* reasoning: if we can establish that $s \models \sigma$ on the hypothetical assumption that we are in another situation (for example, s) then we should be able to conclude that $s \models \sigma$. This is because of the status of $s \models \sigma$: it expresses a proposition, so if it is true it is true everywhere.

We capture the situation-independence of propositions by a separate principle of reasoning which we call the *Hypothetical Term Rule*. If we can argue that the proposition p follows from propositional premises $p_1, \ldots, p_n$ together with the hypothetical assumption that we are in the situation named by the term s then the argument goes through without this last assumption. In other words, no argument involving propositions alone can depend on the current situation. As an example, consider the following derivation:

$$\dfrac{\dfrac{[s \models (\sigma \vee \sigma')]^2 \qquad s}{\sigma \vee \sigma'}{\models}\mathrm{E} \qquad \dfrac{\dfrac{s \qquad [\sigma]^1}{s \models \sigma'}{\models}\mathrm{I}}{s \models \sigma \vee s \models \sigma'}\vee\mathrm{I} \qquad \dfrac{\dfrac{s \qquad [\sigma']^1}{s \models \sigma'}{\models}\mathrm{I}}{s \models \sigma \vee s \models \sigma'}\vee\mathrm{I}}{\dfrac{s \models \sigma \vee s \models \sigma'}{s \models (\sigma \vee \sigma') \to (s \models \sigma \vee s \models \sigma')}{\to}\mathrm{I}^2}\vee\mathrm{E}^1$$

This derivation is a natural deduction proof of the formula $s \models (\sigma \vee \sigma') \to (s \models \sigma \vee s \models \sigma')$ from the single assumption s. But we are permitted to hypothetically assume that we are in the situation named by s and so, by discharging this assumption, we have a proof of $s \models (\sigma \vee \sigma') \to (s \models \sigma \vee s \models \sigma')$ from no assumptions.

When combined with the usual rules of inference for conjunction, disjunction, implication and negation, the three rules discussed above are enough to give a complete calculus, as will be shown in the following sections. There are two more principles of situated reasoning which we wish to capture, but both can be captured by *admissible* rules of the calculus. They come into there own when we consider quantification in Section 4.

The first of these principles concerns the naming of situations. In constructing a rational argument, we suppose that our situated reasoner should be able to refer to her current situation. The choice of name is up to her as long as it does not occur in any of the premises or the conclusion of the argument she is attempting to evaluate. The *Naming Rule* states that if she can argue to the conclusion ϕ from premises $\phi_1, \ldots, \phi_n$ together with the additional premise s then the argument goes through without this additional assumption, so long as s does not occur in any of the formulae $\phi_1, \ldots, \phi_n$ or in ϕ. Without quantification the *Naming Rule* is next to useless because of the severity of this restriction. But in the quantificational calculus it becomes essential.

The final principle we wish to capture concerns the relationship between entailment at the level of propositions and entailment at the level of infons. One can imagine that there might be some disparity between these two levels reflecting the resources available to a situated reasoner. For example, it may be that classical reasoning is appropriate at the level of propositions (since they have absolute truth values) whilst some weaker principles, such as those of intuitionistic or partial logic, are better suited to reasoning at the level of infons. We acknowledge this possibility, but for the moment we will insist on a strict parity between the two levels.

The *Parity Principle* is that an argument to the conclusion σ from premises $\sigma_1, \ldots, \sigma_n$ is valid if and only if there is a valid argument to the conclusion $s \models \sigma$ from premises $s \models \sigma_1, \ldots, s \models \sigma_n$, so long as s does not occur in any of $\sigma_1, \ldots, \sigma_n$ or in σ.

The *Naming Rule* and the *Parity Principle* are inter-derivable, but the latter will be shown to be more powerful for establishing certain meta-logical results.

2 A Sequent Calculus for Situated Reasoning

In the following formal development we will use a language L_0 of formulae (ranged over by ϕ) consisting of *infon symbols* (ranged over by σ) and *situation names* (ranged over by s) together with the combinations formed using the connectives $\wedge, \vee, \rightarrow, \neg$ and new formulae of the form $s : \phi$ which should be read 's is of type ϕ'. The later will be called *atomic propositions.* The class of *propositions* is inductively defined to consist of the atomic propositions and combinations of them.

In contrast to our informal presentation in the previous section, we do not include the symbol '$\models$' in L_0. This is partly for reasons of textual economy and tradition — ':' takes up less space than '$\models$' and the later is traditionally reserved for meta-logical purposes — but also because of the common use of '$\models$' in situation theory where it is restricted to occurring between a situation term and an infon *name.* We do not intend to treat infon names in this paper since a proper treatment would take us outside our remit of developing a calculus for extensional elementary situation theory.[1] A reasonable approximation would be to *define* $s \models \sigma$ to be $s : \sigma$ (where σ is an infon symbol).

A *sequent* is a pair of finite sets Γ, Δ of formulae, written $\Gamma \vdash \Delta$. The formulae in Γ are called *antecedent* formulae and the formulae in Δ are called *succedent*

[1] Equality of infons is usually taken to be intensional and quantification over infons would introduce higher-order aspects to the logic.

formulae. Informally, a sequent holds in the current situation iff one of its succedent formulae is true or one of its antecedent formulae is not true in the situation. A formal semantics for the calculus will be given in the next section.

Our calculus is an extension of a standard Gentzen-style calculus for classical logic. The classical axioms and rules we use are as follows:

$$\frac{}{\phi, \Gamma \vdash \Delta, \phi}\text{Axiom} \qquad \frac{\Gamma \vdash \Delta, \phi \quad \phi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\text{Cut}$$

$$\frac{\phi, \Gamma \vdash \Delta}{\phi \wedge \phi', \Gamma \vdash \Delta}\text{L1}\wedge \qquad \frac{\phi', \Gamma \vdash \Delta}{\phi \wedge \phi', \Gamma \vdash \Delta}\text{L2}\wedge \qquad \frac{\Gamma \vdash \Delta, \phi \quad \Gamma \vdash \Delta, \phi'}{\Gamma \vdash \Delta, \phi \wedge \phi'}\text{R}\wedge$$

$$\frac{\phi, \Gamma \vdash \Delta \quad \phi', \Gamma \vdash \Delta}{\phi \vee \phi', \Gamma \vdash \Delta}\text{L}\vee \qquad \frac{\Gamma \vdash \Delta, \phi}{\Gamma \vdash \Delta, \phi \vee \phi'}\text{R1}\vee \qquad \frac{\Gamma \vdash \Delta, \phi'}{\Gamma \vdash \Delta, \phi \vee \phi'}\text{R2}\vee$$

$$\frac{\Gamma \vdash \Delta, \phi \quad \phi', \Gamma \vdash \Delta}{\phi \to \phi', \Gamma \vdash \Delta}\text{L}\to \qquad \frac{\phi, \Gamma \vdash \Delta, \phi'}{\Gamma \vdash \Delta, \phi \to \phi'}\text{R}\to$$

$$\frac{\Gamma \vdash \Delta, \phi}{\neg\phi, \Gamma \vdash \Delta}\text{L}\neg \qquad \frac{\phi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg\phi}\text{R}\neg$$

The more user-friendly rules of *Weakening* and *Generalized Cut* are admissible rules in this calculus:

$$\frac{\Gamma \vdash \Delta}{\Gamma', \Gamma \vdash \Delta, \Delta'}\text{Weak} \qquad \frac{\Gamma \vdash \Delta, \phi \quad \phi, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{G. Cut}$$

More importantly, a variant of Gentzen's *Hauptsatz* shows that the *Cut* rule can be eliminated from the calculus. In other words, any sequent provable in the above calculus has a proof which does not involve the use of the *Cut* rule. The main significance of this famous result is given by a simple corollary. Noting that the formulae occurring in the premises of each rule (except *Cut*) are all subformulae of formulae occurring in the conclusion of the rule, it is clear that the *Cut*-free calculus has the *Subformula Property*: every formula occurring in any proof of a sequent $\Gamma \vdash \Delta$ is a subformula of a formula in Γ or Δ. In other words, the logic (classical propositional logic, in this case) obeys a principle of "purity of methods" (Girard 1987) or is *analytic* (Smullyan 1968, Fitting 1983); the possible proofs of a sequent being determined entirely by its parts.

To this extremely elegant calculus, we add new rules to reflect the principles of situated reasoning outlined in the previous section. The first two rules are the Gentzen-style left and right rules for ':' which correspond to the elimination and introduction rules presented earlier.

$$\frac{\Gamma \vdash \Delta, s \qquad \phi, \Gamma \vdash \Delta}{s : \phi, \Gamma \vdash \Delta}\text{L:} \qquad\qquad \frac{\Gamma \vdash \Delta, s \qquad \Gamma \vdash \Delta, \phi}{\Gamma \vdash \Delta, s : \phi}\text{R:}$$

With these two new rules the calculus retains the property of the eliminability of *Cut*, the subformula property and the admissibility of *Weakening* and *Generalized Cut*. However, it is not complete with respect to our (presently informal) interpretation. For example, the sequent $s : (\sigma \wedge \sigma') \vdash s : \sigma$ is not provable.

To complete our calculus we need a (slightly generalized) sequent version of the Hypothetical Term Rule. To prove a sequent $\Gamma \vdash \Delta$ consisting entirely of propositions, we can use any situation name s as an additional antecedent formula, i.e., it is enough to prove the sequent $s, \Gamma \vdash \Delta$. The slight generalization amounts to throwing away any non-propositional formulae which occur in the original sequent, viz.,

$$\frac{t, \Gamma \vdash \Delta}{\Gamma', \Gamma \vdash \Delta, \Delta'}\text{Term } (\Gamma, \Delta \text{ propositions})$$

The calculus obtained by adding the rules L:, R: and *Term* to the rules for classical logic will be called S_0. The addition of the *Term* rule, though necessary for completeness, presents certain problems for the proof theory. *Weakening* and *Generalized Cut* are still admissible — our admittance of Γ' and Δ' in the conclusion ensures this — but *Cut* elimination and the Subformula Property are lost. We will return to this problem in Section 5.

3 Semantics

A model $\mathbf{m}$ for L_0 is defined to be a pair $\langle m, [.]^m \rangle$ where m is a set whose elements are thought of as *situations* and $[.]^m$ is an interpretation function assigning an *element* of m to each situation name and a *subset* of m to each infon symbol. Intuitively, $[s]$ is the situation named by s and $[\sigma]$ is the set of situations which support σ. The superscripted m of $[.]^m$ will often be dropped, as it was in the last sentence. Truth in a model $\mathbf{m}$ is defined relative to each situation $i \in m$ by the conditions

1. s is true in $\mathbf{m}$ at i iff $[s]^m = i$
2. σ is true in $\mathbf{m}$ at i iff $i \in [\sigma]^m$

3. $s : \phi$ is true in $\mathbf{m}$ at i iff ϕ is true in $\mathbf{m}$ at $[s]^m$

extended inductively to combinations using $\wedge, \vee, \rightarrow$ and $\neg$ in the usual way. Note that the truth of a *proposition* in $\mathbf{m}$ at i is entirely independent of the situation i. A sequent $\Gamma \vdash \Delta$ is said to *hold* at i in $\mathbf{m}$ iff one of its succedents is true at i or one of its antecedents is not true at i. A sequent is ***valid in* m** iff it holds at every situation in $\mathbf{m}$, and L_0 valid (or just plain 'valid') if it is valid in every L_0 model. Note that $\vdash s : \phi$ is valid in $\mathbf{m}$ iff $[s]^m$ supports ϕ.

Theorem 1 Correctness and Completeness. An L_0 sequent is valid iff it is provable in the calculus S_0.

PROOF: The correctness of S_0 with respect to the above interpretation of L_0 follows from the correctness of each sequent rule, which can be established by inspection. Only the *Term* rule deserves any comment. If Γ and Δ are sets of propositions and $\Gamma', \Gamma \vdash \Delta, \Delta'$ is *not* valid then there is an L_0 model $\mathbf{m}$ and a situation $i \in m$ which supports all the Γ', Γ but none of the Δ, Δ'. Since the Γ and Δ are all propositions, it follows that *every* situation in m supports all the Γ and none of the Δ. In particular, $[s]^m$ supports all the Γ and none of the Δ. But $[s]^m$ also supports s, and so the sequent $s, \Gamma \vdash \Delta$ does not hold at $[s]^m$ in $\mathbf{m}$. Hence, $s, \Gamma \vdash \Delta$ is not valid and, from the contrapositive, the *Term* rule is proved correct. Completeness follows in a standard way from the following Model Existence Theorem. □

Theorem 2 Model Existence. If Δ is an S_0-consistent set of L_0 formulae (i.e., there in no $\Gamma \subseteq \Delta$ such that $\Gamma \vdash$ is provable in S_0) then there is an L_0 model containing a situation at which every formula in Δ is true.

PROOF: Let L_0^+ be the language obtained by extending L_0 with a new situation name, s_Δ. Let Δ^+ be a maximally S_0-consistent set of L_0^+ formulae containing s_Δ and every formula in Δ. The existence of such a set can be established by a standard Lindenbaum construction. It is easy to show (using the *Cut* rule and the maximal consistency of Δ^+) that Δ^+ is logically closed, i.e., if $\Gamma \subseteq \Delta^+$ and the sequent $\Gamma \vdash \Gamma'$ is provable in S_0 then Δ^+ contains at least one of the formulae in Γ'. We will use this fact throughout the proof.

First we construct an L_0^+ model $\mathbf{m}_\Delta$ as follows. For each situation name s, let

$$[s]^{m_\Delta} = \{\phi \mid s : \phi \in \Delta^+\}$$

Let m_Δ be the set consisting of the sets $[s]$, for each L_0^+ situation name s. For each L_0^+ infon symbol σ, let

$$[\sigma]^{m_\Delta} = \{i \in m_\Delta \mid \sigma \in i\}$$

We want to show that for all i in m_Δ and for each L_0^+ formula ϕ,

ϕ is true at i in $\mathbf{m}_\Delta$ iff $\phi \in i$

Noting first that for each $i \in m_\Delta$ there is a situation name s_i such that $[s_i] = i$ we proceed by induction on the structure of ϕ.

1. s is true at i iff $[s] = i$. Now $\vdash s : s$ is provable in S_0 so $s : s$ is in Δ^+ by logical closure and so $s \in [s]$. So if $[s] = i$ then $s \in i$. Conversely, if $s \in i$ then, since $i = [s_i]$, $s_i : i$ is in Δ^+. But, for each L_0 formula ϕ, the sequents $s_i : s, s_i : \phi \vdash s : \phi$ and $s_i : s, s : \phi \vdash s_i : \phi$ are provable in S_0 and so, by logical closure again, $s_i : \phi$ is in Δ^+ iff $s : \phi$ is. Consequently, $i = [s_i] = [s]$.
2. σ is true at i iff $i \in [\sigma]$ iff $\sigma \in i$.
3. $s : \phi$ is true at i iff ϕ is true at $[s]$. By the inductive hypothesis ϕ is true at $[s]$ iff $\phi \in [s]$. By definition, $\phi \in [s]$ iff $s : \phi$ is in Δ^+. Since the sequents $s : \phi \vdash s_i : (s : \phi)$ and $s_i : (s : \phi) \vdash s : \phi$ are provable in S_0 and Δ^+ is logically closed, $s : \phi$ is in Δ^+ iff $s_i : (s : \phi)$ is. Finally, $s_i : (s : \phi)$ is in Δ^+ iff $s : \phi$ is in $[s_i] = i$.

The remaining steps in the inductive argument are for combinations of L_0 formulae formed with the symbols $\wedge, \vee, \rightarrow$ and $\neg$. In each case the proof is similar to that of 3. (above) depending, respectively, on the S_0 provability of the following sequents:

4. $s_i : (\phi \wedge \phi') \vdash s_i : \phi$ and $s_i : (\phi \wedge \phi') \vdash s_i : \phi'$ and $s_i : \phi, s_i : \phi' \vdash s_i : (\phi \wedge \phi')$,
5. $s_i : (\phi \vee \phi') \vdash s_i : \phi, s_i : \phi'$ and $s_i : \phi \vdash s_i : (\phi \vee \phi')$ and $s_i : \phi' \vdash s_i : (\phi \vee \phi')$,
6. $s_i : (\phi \rightarrow \phi'), s_i : \phi \vdash s_i : \phi'$ and $\vdash s_i : (\phi \rightarrow \phi'), s_i : \phi$ and $s_i : \phi' \vdash s_i : (\phi \rightarrow \phi')$, and
7. $s_i : \neg\phi, s_i : \phi \vdash$ and $\vdash s_i : \neg\phi, s_i : \phi$.

Finally, note that, since $s_\Delta \in \Delta^+$, $[s_\Delta] = \Delta^+$ and so every formula in Δ is true at $[s_\Delta]$ in $\mathbf{m}_\Delta$. □

Theorem 3 Admissibility of Naming and Parity. Given a set X of formulae, let $s : X$ be the set of formulae $\{s : x \mid x \in X\}$. If Γ and Δ are sets of L_0 formulae which do not contain occurrence of the situation name s and any one of the sequents

$$s, \Gamma \vdash \Delta, \qquad s : \Gamma \vdash s : \Delta \qquad \text{or} \qquad \Gamma \vdash \Delta$$

is valid then so are the other two.

Proof: We show, in turn, that *in*validity is transmitted from each sequent to the next. If $s, \Gamma \vdash \Delta$ is not valid then there is an L_0 model $\mathbf{m}$ with a situation $i \in m$ at which both s and each formula in Γ is true and at which each formula in Δ is not true. Thus $[s] = i$ and so, at i, each formula in $s : \Gamma$ is true and each formula in $s : \Delta$ is not true. So $s : \Gamma \vdash s : \Delta$ does not hold at i and so is not valid.

If $s : \Gamma \vdash s : \Delta$ is not valid then there is an L_0 model $\mathbf{m}$ with a situation $i \in m$ at which each formula in $s : \Gamma$ is true and at which each formula in $s : \Delta$ is not true. At $[s]$ each formula of Γ is true and each formula in Δ is not true. So $\Gamma \vdash \Delta$ does not hold at s and so is not valid. So far we have not not use the assumption that s does not occur in Γ, Δ so the implication of validity from $s, \Gamma \vdash \Delta$ to $s : \Gamma \vdash s : \Delta$ to $\Gamma \vdash \Delta$ holds for an arbitrary situation name, s.

Now let L_0^- be the language obtained by removing from L_0 any formula containing an occurrence of the name s. Suppose that the sequent $\Gamma \vdash \Delta$ is not L_0 valid. Since any L_0 model can be changed into an L_0^- model by removing s from the domain of the interpretation function, it is clear from the definition of truth-at-a-situation that $\Gamma \vdash \Delta$ (an L_0^- sequent) is not L_0^- valid. So there must be an L_0^- model $\mathbf{m}^-$ with a situation i at which none of the formulae in Δ but all of the formula in Γ are true. Now let $\mathbf{m}$ be the L_0 model obtained from $\mathbf{m}^-$ by setting $[s] = i$. As s is clearly true at i in $\mathbf{m}$ together with all the formulae in Γ and none of the formulae in Δ, the sequent $s, \Gamma \vdash \Delta$ fails to hold at i and so is not valid. □

4 Quantification and General Terms

The addition of first-order quantification to our calculus is relatively straightforward. We extend the set of situation names to include a (countably infinite) set of new symbols called *parameters*. We also need a (countably infinite) set of new symbols called *variables*. The situation names together with the parameters (but not the variables) will be called *terms*. We use the meta-variables a and x to range over parameters and variables, respectively. The language L_1 is then obtained by inductively extending L_0 to include formulae of the form $\forall x \phi_x^a$ and $\exists x \phi_x^a$ for each formula ϕ which *contains no occurrences of the variable* x, and where ϕ_x^a is the result of replacing all occurrences of a in ϕ by x.

This use of parameters is a standard trick for avoiding the problem of variable "clashes" and the need for a complex definition of substitution. Note that all variables in L_1 formulae occur bound. If an L_1 formula contains no parameters it is *non-parametric*; otherwise it is *parametric*. The inductive definition of L_0 propositions is extended a definition of L_1 *propositions* by including formulae of the form $\forall x \phi_x^a$ and $\exists x \phi_x^a$ when ϕ_x^a is an L_1 proposition.

The quantifiers are understood as ranging over situations. For example, the L_1 formula $\forall x(x : \phi \rightarrow x : \psi)$ is interpreted to mean that every situation of type ϕ is also of type ψ. To capture this meaning in our calculus we must add the usual rules for quantification, i.e.,

$$\frac{\sigma_t^x, \Gamma \vdash \Delta}{\forall x.\sigma, \Gamma \vdash \Delta}\text{L}\forall \qquad\qquad \frac{\Gamma \vdash \Delta, \sigma}{\Gamma \vdash \Delta, \forall x.\sigma_x^a}\text{R}\forall\ (a \text{ new})$$

$$\frac{\sigma, \Gamma \vdash \Delta}{\exists x.\sigma_x^a, \Gamma \vdash \Delta}\text{L}\exists\ (a \text{ new}) \qquad\qquad \frac{\Gamma \vdash \Delta, \sigma_t^x}{\Gamma \vdash \Delta, \exists x.\sigma}\text{R}\exists$$

where the restriction to a's being a *new* parameter means that a must not occur in any formula below the line.

The addition of quantifier rules is not quite enough to give a complete calculus, since — for example — the intuitively valid sequent $\sigma \vdash \exists x(x : \sigma)$ is not provable. We also need the *Naming Rule*:

$$\dfrac{a, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\text{Name } (a \text{ new})$$

The calculus for L_1 obtained by extending S_0 with the rules R$\forall$, L$\forall$, R$\exists$, L$\exists$ and *Name* will be called S_1. The restriction on the *Term* rule is weakened to allow L_1 propositions below the line. The following result gives some hints as to how to provide a semantic interpretation for L_1 for which S_1 is a correct calculus.

Proposition 4 The following sequents are all provable in S_1:

1. $\vdash \forall x(x : (\phi \wedge \phi') \leftrightarrow x : \phi \wedge x : \phi')$
2. $\vdash \forall x(x : (\phi \vee \phi') \leftrightarrow x : \phi \vee x : \phi')$
3. $\vdash \forall x(x : (\phi \rightarrow \phi') \leftrightarrow x : \phi \rightarrow x : \phi')$
4. $\vdash \forall x(x : \neg\phi \leftrightarrow \neg x : \phi)$
5. $\vdash \forall x(x : \forall x'\phi \leftrightarrow \forall x'(x : \phi))$
6. $\vdash \forall x(x : \exists x'\phi \leftrightarrow \exists x'(x : \phi))$

PROOF: As an example of S_1 in action we will give a proof of sequent 4. There are S_1 proofs with a similar structure for each of the other sequents (although care must be taken in the order of application of the quantifier rules in the proofs of sequents 6. and 7.).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a, a : \neg\sigma \vdash a}\text{Axiom} \quad \dfrac{\dfrac{}{a, \sigma \vdash a}\text{Axiom} \quad \dfrac{\dfrac{}{a, \sigma \vdash \sigma}\text{Axiom}}{a, \sigma, \neg\sigma \vdash}\text{L}\neg}{a, \sigma, a : \neg\sigma \vdash}\text{L:}}{a, a : \sigma, a : \neg\sigma \vdash}\text{L:}}{a : \sigma, a : \neg\sigma \vdash}\text{Term}}{a : \neg\sigma \vdash \neg(a : \sigma)}\text{R}\neg}{\vdash a : \neg\sigma \rightarrow \neg(a : \sigma)}\text{R}\rightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash a : \neg\sigma, a}\text{Axiom} \quad \dfrac{\dfrac{}{a \vdash \sigma, a}\text{Axiom} \quad \dfrac{\dfrac{}{\sigma, a \vdash \sigma}\text{Axiom}}{a \vdash \sigma, \neg\sigma}\text{R}\neg}{a \vdash \sigma, a : \neg\sigma}\text{R:}}{a \vdash a : \sigma, a : \neg\sigma}\text{R:}}{\vdash a : \sigma, a : \neg\sigma}\text{Term}}{\neg(a : \sigma) \vdash a : \neg\sigma}\text{L}\neg}{\vdash \neg(a : \sigma) \rightarrow a : \neg\sigma}\text{R}\rightarrow}{\dfrac{\vdash a : \neg\sigma \leftrightarrow \neg(a : \sigma)}{\vdash \forall x(x : \neg\sigma \leftrightarrow \neg(x : \sigma))}\text{R}\forall}\text{R}\wedge$$

$\square$

The models for L_1 are just the same as the models for L_0 except that we need two auxiliary functions to define truth and sequent validity. The first is an *anchor* which maps parameters to situations. The second in an *assignment* which maps variables to situations.

As remarked earlier, L_1 formulae do not have free variables, but in the following definition we will need the notion of a *pseudo-formula* of L_1, which is basically like an L_1 formulae but with free variables. More precisely, an L_1 pseudo-formula is anything that results from the uniform substitution of variables $x_1, \ldots, x_n$ for parameters $a_1, \ldots, a_n$ in an L_1 formula (for arbitrary n).

Given a model $\mathbf{m}$, an anchor α, an assignment g and a situation $s \in m$, we define the property of 'being true at i in $\mathbf{m}$ with respect to α and g' for L_1 pseudo-formulae in the same way as we defined the property of 'being true at i in $\mathbf{m}$' for L_0 formulae. In addition, we have two extra base clauses:

1.1. a is true at i in $\mathbf{m}$ with respect to α and g iff $\alpha(a) = i$, and

1.2. x is true at i in $\mathbf{m}$ with respect to α and g iff $g(x) = i$;

and the expected inductive clauses for quantifiers:

8. $\forall x\phi$ is true at i in $\mathbf{m}$ with respect to α and g iff for every assignment h such that $g < x > h$, ϕ is true at i in $\mathbf{m}$ with respect to α and h, and

9. $\exists x\phi$ is true at i in $\mathbf{m}$ with respect to α and g iff there is an assignment h such that $g < x > h$, ϕ is true at i in $\mathbf{m}$ with respect to α and h,

where $g < x > h$ iff g and h are assignments which differ in at most the value they assign to the variable x.

An L_1 formulae is true at i in $\mathbf{m}$ with respect to α just in case it is true at i in $\mathbf{m}$ with respect to α and every — equivalently, some — assignment. A sequent *holds* at i in $\mathbf{m}$ with respect to α iff one its antecedents is not true at i (with respect to α) or one of its succedents is true at i (with respect to α). A sequent is *valid in* $\mathbf{m}$ iff it holds at every situation in $\mathbf{m}$ with respect to every anchor, and L_1 *valid* (or just plain 'valid') if it is valid in every L_1 model.

Theorem 5 Correctness and Completeness. An L_1 sequent is valid iff it is provable in the calculus S_1.

PROOF: The correctness of S_1 with respect to L_1 validity follows from the correctness of S_0 with respect to L_O validity, the easily checked correctness of the quantifier rules and the correctness of the *Naming Rule* which was essentially proved in Theorem 3. Completeness follows, as usual, from the Model Existence Theorem. □

Theorem 6 Model Existence.

1. If Δ is an S_1-consistent set of non-parametric L_1 formulae then there is an L_1 model containing a situation at which every formula in Δ is true.

2. If Δ is an S_1-consistent set of L_1 formulae then there is an L_1 model containing a situation i and an anchor α such that every formula in Δ is true at i with respect to α.

PROOF: The second part is a simple generalization of the first, so we just prove the first part. Extend Δ to a maximally S_1-consistent set Δ^+ of (possibly parametric) L_1 formulae which obeys the *witnessing condition*, i.e., for each L_1 formula of the form $\exists x\phi$ there is a parameter a such that Δ contains the formula

$$\exists x\phi \to \phi^x_a.$$

The existence of such a set can be established by a standard Lindenbaum construction. Again, by the *Cut* rule, it is logically closed. Let L_0^+ be the extension of L_0 to include the parameters of L_1 as new situation names. From the L_0^+ model $\mathbf{m}_{\Delta^+}$ — as constructed in the proof of Theorem 2 — we can read off an L_1 model $\mathbf{m}$ (by restricting $[.]^{\mathbf{m}_{\Delta^+}}$ to the situation names and infon symbols of L_0) and an anchor α, which is defined to anchor each parameter a to the situation $[a]^{\mathbf{m}_{\Delta^+}}$. As before, we need to prove that, for each situation i in m, a (non-parametric) formula is true at i iff it is contained in i. This will be done below. Since $\vdash \exists xx$ is S_1-provable and there is a parameter a_Δ such that the formula $\exists xx \to a$ is in Δ^+, it follows from the logical closure of Δ^+ that a_Δ is in Δ^+. From the S_1-provability of sequents of the form $a_\Delta, \phi \vdash a_\Delta : \phi$ we can conclude that every formula in Δ is contained in the situation $\alpha(a_\Delta)$. The theorem therefore follows from the following key lemma:

Lemma 7 For each L_1 pseudo-formula ϕ (with free variables $x_1, \dots, x_n$), each i in m and each assignment g, ϕ is true at i in $\mathbf{m}$ with respect to α and g just in case there are terms $t_1, \dots, t_n$ such that $t_1 \in g(x_1), \dots, t_n \in g(x_n)$ and $\phi^{x_1 \dots x_n}_{t_1 \dots t_n} \in i$.

Recall that variables are not considered to be terms of L_1. The proof of the lemma is (again) by induction on the structure of ϕ. Cases 1. to 7. are almost identical to those in the proof of Theorem 2 so we will omit them. It remains to provide proofs for the two new base cases and for the cases where ϕ is a quantified pseudo-formula. Since the latter are very similar, we will only prove the case where ϕ is existentially quantified.

1. a is true at i (w.r.t. α,g) iff $\alpha(a) = i$. By definition of α, $\alpha(a) = [a]$ and (as we have seen before) $[a] = i$ iff $a \in i$.

2. x is true at i (w.r.t. α,g) iff $g(x) = i$. The construction of $\mathbf{m}$ ensures that $g(x) = [t]$ for some term t, but $x^x_t = t$ and $[t] = i$ iff $t \in i$.

9. If $\exists x\phi$ is true at i (w.r.t. α,g) then there is an assignment h such that $g < x > h$ and ϕ is true at i (w.r.t. α,h). Suppose ϕ has free variable $x_1, \dots x_n, x$. By the inductive hypothesis, there are terms $t_1, \dots, t_n, t$ such that $t_1 \in h(x_1), \dots t_n \in h(x_n), t \in h(x)$ and $\phi^{x_1 \dots x_n, x}_{t_1 \dots t_n, t} \in i$. Hence the formula $s_i : \phi^{x_1 \dots x_n, x}_{t_1 \dots t_n, t}$ is in Δ^+. By the S_1-provability of the sequent

$$s_i : \phi^{x_1 \dots x_n, x}_{t_1 \dots t_n, t} \vdash s_i : \exists x\phi^{x_1 \dots x_n}_{t_1 \dots t_n}$$

and the logical closure of Δ^+ it follows that the formula $\exists x\phi^{x_1 \dots x_n}_{t_1 \dots t_n}$ is in i. Since $g < x > h$, $t_1 \in g(x_1), \dots t_n \in g(x_n)$ and we are done.

Conversely, if there are terms $t_1, \dots t_n$ such that $t_1 \in g(x_1), \dots t_n \in g(x_n)$ and $\exists x\phi^{x_1 \dots x_n}_{t_1 \dots t_n}$ is in i then $s_i : \exists x\phi^{x_1 \dots x_n}_{t_1 \dots t_n}$ is in Δ^+. The witnessing condition ensures that there is a parameter a such that the formula $\exists x s_i : \phi^{x_1 \ldots x_n}_{t_1 \ldots t_n} \to s_i : \phi^{x_1 \ldots x_n, x}_{t_1 \ldots t_n, a}$ is also in Δ^+. By the S_1-provability of the sequent

$$s_i : \exists x \phi^{x_1 \ldots x_n}_{t_1 \ldots t_n} \vdash \exists x s_i : \phi^{x_1 \ldots x_n}_{t_1 \ldots t_n}$$

and the logical closure of Δ^+, it follows that $s_i : \phi^{x_1 \ldots x_n, x}_{t_1 \ldots t_n, a}$ is in Δ^+ and so $\phi^{x_1 \ldots x_n, x}_{t_1 \ldots t_n, a}$ is in i. Let h' be the assignment which differs from g only in assigning a to x and note that $t_1 \in h'(x_1), \ldots, t_n \in h'(x_n), a \in h'(x)$. By the inductive hypothesis, it follows that ϕ is true at i (w.r.t. α, h'). Noting that $g < x > h'$, we can conclude $\exists x \phi$ is true at i (w.r.t. α, g).

□

The languages L_0 and L_1 are not particularly useful for applications. With only atomic "infon symbols" to work with little can be said. What we need is the ability to express predicated information of the form $R(t_1, \ldots, t_n)$ and refer to situations with complex terms of the form $f(t_1, \ldots, t_n)$. In fact, we will go one stage further and allow expressions of the form $R(\phi_1, \ldots, \phi_n)$ and $f(\phi_1, \ldots, \phi_n)$ for arbitrary formulae $\phi_1, \ldots, \phi_n$. What could such expressions possibly mean? In the unary case they have the following interpretation:

> $R(\phi)$ is true at i iff there is some j that has the property R from the point of view of i and ϕ is true at j
>
> $f(\phi)$ is true at i iff i is the f-image of some j at which ϕ is true.

Some plausible natural-language-like examples include 'right(car-crash)', meaning that a car-crash has occurred to my right, and 'end-of(scrabble-game)', meaning that we are at (or, rather, the current situation is) the end of a game of scrabble.

This generalization allows us to define a streamlined formal language L_2 in which formulae are freely generated from the parameters, some constants (of arities ≥ 0) and the logical symbols $\wedge, \vee, \to, \neg$ and :. The formation of quantified formulae follows the method used in the definition of L_1. The terms of L_2 are defined by picking a subset of the constants to be *function symbols* and allowing free generation over the set of parameters. The constants that are not function symbols are called *relation symbols*. It is technically convenient to regard ':' as both a logical symbol *and* a binary relation symbol. [2]

To the calculus S_1, appropriately extended to the new language, we need to add two more logical rules to cope with occurrences of non-terms inside atomic formulae. These are:

[2] Note that expressions of the form $\phi : t$ and $\phi : \phi'$ are formulae of L_2. The former means exactly the same as $t : \phi$ and the latter means that there is some situation at which ϕ and ϕ' are both true.

$$\frac{a:\phi, F(\ldots_i\, a \ldots), \Gamma \vdash \Delta}{F(\ldots_i\, \phi \ldots), \Gamma \vdash \Delta}\mathrm{L}\mathcal{F}\ (a\ \text{new}) \qquad\qquad \frac{\Gamma \vdash \Delta, t:\phi \qquad \Gamma \vdash \Delta, F(\ldots_i\, t \ldots)}{\Gamma \vdash \Delta, F(\ldots_i\, \phi \ldots)}\mathrm{R}\mathcal{F}$$

where F is a constant of arity n and $F(\ldots_i\, \mu \ldots)$ is schematic over $F(\phi_1, \ldots, \phi_n)$ with μ substituted in for ϕ_i. By the condition 'a new' we mean that a is a parameter which does not occur below the line. The resulting calculus will be called S_2.

To give a formal semantics for L_2 we extend the interpretation function $[.]$ of each L_1 model $\mathbf{m}$ by assigning a function $[f] : m^n \to m$ to each n-ary function symbol f and a relation $[R] \subseteq m^{n+1}$ to each n-ary relation symbol R. The truth conditions for atomic formulae are as follows:

1. $f(\phi_1, \ldots, \phi_n)$ is true at i (w.r.t. α, g) iff there are $j_1, \ldots, j_n \in m$ such that $i = [f](j_1, \ldots, j_n)$ and $\phi_1, \ldots, \phi_n$ are true (w.r.t. α, g) at $j_1, \ldots, j_n$, respectively, and

2. $R(\phi_1, \ldots, \phi_n)$ is true at i (w.r.t. α, g) iff there are $j_1, \ldots, j_n \in m$ such that $\langle i, j_1, \ldots, j_n \rangle \in [R]$ and $\phi_1, \ldots, \phi_n$ are true (w.r.t. α, g) at $j_1, \ldots, j_n$, respectively.

Sequents of L_2 formulae can hold at a situation, be valid in a model, or be L_2 valid in exactly the same way that sequents of L_1 formulae can.

Theorem 8 Correctness and Completeness. An L_2 sequent is provable in S_2 iff it is L_2 valid.

PROOF: As before, the new rules must be checked for correctness and the Model Existence Theorem does the rest. □

Theorem 9 Model Existence.

1. If Δ is an S_2-consistent set of non-parametric L_2 formulae then there is an L_2 model containing a situation at which every formula in Δ is true.

2. If Δ is an S_2-consistent set of L_2 formulae then there is an L_2 model containing a situation i and an anchor α such that every formula in Δ is true at i with respect to α.

PROOF: As for L_1. We have to check that the obvious syntactic definition of $[f]$ in the model is well-defined and that the key lemma applies in the atomic cases. In the latter connection, note the S_2-provability of the sequent

$$\vdash \forall x(x : F(\phi_1, \ldots, \phi_n) \leftrightarrow \exists x_1 \ldots x_n(x_1 : \phi_1 \wedge \ldots x_n : \phi_n \wedge F(x_1, \ldots, x_n))).$$

□

5 Parity Rules and the Elimination of Cut

We have seen in the previous section that S_0 obeys the *Parity Principle*, but the argument was a semantic one. It will prove insightful to look at the proof theoretic grounds for this result. The *Parity Principle* can be expressed proof-theoretically as the derivability of rules

$$\frac{s : \Gamma \vdash s : \Delta}{\Gamma \vdash \Delta}\text{UP} \qquad\qquad \frac{\Gamma \vdash \Delta}{s : \Gamma \vdash s : \Delta}\text{DP}$$

of *Upward Parity* and *Downward Parity.*

Theorem 10 *Downward Parity* can be derived in S_0 + *Weak.*

Proof: Suppose Γ and Δ are the finite sets $\{\gamma_1, \ldots, \gamma_n\}$ and $\{\delta_1, \ldots, \delta_m\}$ respectively, and let $\bigwedge \Gamma$ and $\bigvee \Delta$ denote the formulae $\gamma_1 \wedge \ldots \wedge \gamma_n$ and $\delta_1 \wedge \ldots \wedge \delta_m$. Any proof of the sequent $\Gamma \vdash \Delta$ can be extended to a proof of $\bigwedge \Gamma \vdash \bigvee \Delta$ by n applications of L$\wedge$ and m applications of R$\vee$. This can then be extended to a proof of $s : \bigwedge \Gamma \vdash s : \bigvee \Delta$ in the following way:

$$\dfrac{\dfrac{\dfrac{}{s, s : \bigwedge \Gamma \vdash s}\text{Axiom} \qquad \dfrac{\dfrac{}{s \vdash \bigvee \Delta, s}\text{Axiom} \qquad \dfrac{\bigwedge \Gamma \vdash \bigvee \Delta}{s, \bigwedge \Gamma \vdash \bigvee \Delta}\text{Weak}}{s, s : \bigwedge \Gamma \vdash \bigvee \Delta}\text{L:}}{s, s : \bigwedge \Gamma \vdash s : \bigvee \Delta}\text{R:}}{s : \bigwedge \Gamma \vdash s : \bigvee \Delta}\text{Term}$$

For any formulae ϕ and ψ, the sequents $s : (\phi \vee \psi) \vdash s : \phi, s : \psi$ and $s : \phi, s : \psi \vdash s : (\phi \wedge \psi)$ are provable in S_0 and so with $m + n$ applications of the *Cut* rule we can obtain a proof of $s : \Gamma \vdash s : \Delta$. □

Theorem 11 *Upward Parity* and the *Naming Rule* are inter-derivable in S_0 + *Weak* + *G.Cut.*

Proof: Suppose Γ and Δ are the finite sets $\{\gamma_1, \ldots, \gamma_n\}$ and $\{\delta_1, \ldots, \delta_m\}$ respectively. We show that an S_0 proof of $s : \Gamma \vdash s : \Delta$ can be turned into an S_0 proof of $s, \Gamma \vdash \Delta$ and vice versa.

If the sequent $s : \Gamma \vdash s : \Delta$ is provable in S_0 then an application of *Weakening* yields a proof of the sequent $s, s : \Gamma \vdash s : \Delta$. For any formula ϕ, the sequents $s, \phi \vdash s : \phi$ and $s, s : \phi \vdash \phi$ are S_0 provable, so by repeating the following applications of the (generalized) *Cut* rule

$$\frac{s, \gamma_1 \vdash s : \gamma_1 \qquad s, s : \Gamma \vdash s : \Delta}{s, \gamma_1, s : (\Gamma - \{\gamma_1\}) \vdash s : \Delta}\text{G.Cut} \qquad\qquad \frac{s, s : \Gamma \vdash s : \Delta \qquad s, s : \delta_1 \vdash \delta_1}{s, s : \Gamma \vdash s : (\Delta - \{\delta_1\}), \delta_1}\text{G.Cut}$$

for each of $\gamma_2, \ldots, \gamma_n$ and $\delta_2, \ldots, \delta_m$ we can obtain a proof of $s, \Gamma \vdash \Delta$.

Conversely, starting with a proof of $s, \Gamma \vdash \Delta$ we can get a proof of $s, s : \Gamma \vdash s : \Delta$ by applying the L: rule n times and the R: rule m times. The sequent $s : \Gamma \vdash s : \Delta$ then follows by one application of the *Term* rule. The restriction on the *Term* rule is met since all formulae in $s : \Gamma$ and $s : \Delta$ are propositions. $\square$

We would like to be able to prove a third theorem — that the *Naming Rule* is derivable in S_0 — by proof-theoretic means. Together with the above two theorems, we would then have a proof-theoretic analysis of the *Parity Principle* in S_0. Our intuition is that in S_0 the *Naming Rule* is useless. A proof of $s, \Gamma \vdash \Delta$ cannot have really *used* the antecedent s unless s also occurs somewhere in Γ or Δ. To verify this intuition we could try transforming the proof of $s, \Gamma \vdash \Delta$ by simply removing all formulae containing occurrences of the term s. Would the resulting structure be an S_0 proof?

Unfortunately the answer is no, not in general. If our calculus was *analytic* then the we could complete the argument. Proceeding hypothetically, we could firstly show that the term s could only occur on the left-hand side of sequents in the original proof. An inspection of the rules shows that the only way an s could occur of the right-hand side would be if there was an application of one of the rules L: or R:. Any such application would contain an occurrence of a formula of the form $s : \phi$ but this could not be since, if the calculus were analytic, $s : \phi$ would have to occur as a subformula of the end-sequent $s, \Gamma \vdash \Delta$. [3] With this established — albeit hypothetically — the only rule application that could possibly be disrupted by the removal of formulae containing s would be applications of the *Axiom* rule of the form $s, \Gamma' \vdash \Delta', s$. But no such applications could have occurred in the original proof, since s could not have occurred on the right-hand side of a sequent.

So much for hypothetical proofs! What we have established is a likely connection between the derivability of the *Parity Principle* and the analyticity of the calculus. Why then is S_0 not analytic?

There are two reasons, one trivial and one deep. Firstly, the *Term* rule introduces a formula — the hypothetical term — above the line which does not occur below the line. The consequent potential breach of the subformula principle can be avoided by ensuring that applications of the *Term* rule only occur directly beneath applications of one of the rules L: or R: as in the following proof fragments:

[3] This step in our hypothetical argument is not valid for the calculus S_1. For example, the formula $s : \sigma$ is a subformula of $\exists x(x : \sigma)$, which *could* occur in the end-sequent.

$$\dfrac{\dfrac{\dfrac{}{s,\Gamma \vdash \Delta, s}\text{Axiom} \qquad s,\phi,\Gamma \vdash \Delta}{s, s:\phi, \Gamma \vdash \Delta}\text{L:}}{s:\phi,\Gamma \vdash \Delta}\text{Term} \qquad\qquad \dfrac{\dfrac{\dfrac{}{s,\Gamma \vdash \Delta, s}\text{Axiom} \qquad s,\Gamma \vdash \Delta,\phi}{s,\Gamma \vdash \Delta, s:\phi}\text{R:}}{\Gamma \vdash \Delta, s:\phi}\text{Term}$$

An examination of the S_0 and S_1 proofs we have given — and of those we have assumed — reveals that the *Term* rule *can* be restricted in this way. In other words, we can conduct all proofs in the calculus obtained from S_0 by replacing the *Term* rule by the rules

$$\dfrac{s,\phi\Gamma \vdash \Delta}{s:\phi,\Gamma \vdash \Delta}\text{TL:} \qquad\qquad \dfrac{s,\Gamma \vdash \Delta,\phi}{\Gamma \vdash \Delta, s:\phi}\text{TR:}$$

where the sets Γ and Δ are made up of propositions. These rules clearly obey the subformula principle. [4]

The second reason for the lack of analyticity is the *Cut* rule. An application of the *Cut* rule can introduce a completely arbitrary formula into the proof. What we need is a *Cut* elimination theorem. Following Gentzen (1934) we see that the logical rules, including L: and R:, all behave in the appropriate way and *Cut*s can be systematically replaced by *Cut*s with a *Cut* formula of lower complexity. For example, *Cut*s arising from the new logical rules can be reduced in complexity by the following transformations:[5]

$$\dfrac{\dfrac{\Gamma \vdash \Delta, s \qquad \Gamma \vdash \Delta,\phi}{\Gamma \vdash \Delta, s:\phi}\text{R:} \quad \dfrac{\Gamma \vdash \Delta, s \qquad \phi,\Gamma \vdash \Delta}{s:\phi,\Gamma \vdash \Delta}\text{L:}}{\Gamma \vdash \Delta}\text{Cut} \qquad \rightsquigarrow \qquad \dfrac{\Gamma \vdash \Delta,\phi \qquad \phi,\Gamma \vdash \Delta}{\Gamma \vdash \Delta}\text{Cut}$$

---

[4] Note that we must retain the rules L: and R:. Without them we would be at a loss even to prove the sequent $s,\sigma \vdash s:\sigma$.

[5] The *Weakening* steps in this (and subsequent) transformations are not strictly applications of the *Weak* rule. *Weak* is not a rule of any of our calculi, although it is admissible in all of them. Read the (apparent) applications of *Weak* as *transformations* of the proof. For eample, the proof with end-sequent $\Gamma \vdash \Delta, t:\phi$ is transformed to give a proof with end-sequent $F(\ldots_i t \ldots), \Gamma \vdash \Delta, t:\phi$.

$$\dfrac{\dfrac{\Gamma\vdash\Delta, t:\phi \qquad \Gamma\vdash\Delta, F(\ldots_i t\ldots)}{\Gamma\vdash\Delta, F(\ldots_i \phi\ldots)}\mathrm{R}\mathcal{F} \qquad \dfrac{a:\phi, F(\ldots_i a\ldots),\Gamma\vdash\Delta}{F(\ldots_i\phi\ldots),\Gamma\vdash\Delta}\mathrm{L}\mathcal{F}}{\Gamma\vdash\Delta}\mathrm{Cut} \qquad \leadsto$$

$$\dfrac{\Gamma\vdash\Delta,F(\ldots_i t\ldots) \qquad \dfrac{\dfrac{\Gamma\vdash\Delta,t:\phi}{F(\ldots_i t\ldots),\Gamma\vdash\Delta,t:\phi}\mathrm{Weak} \qquad \dfrac{a:\phi,F(\ldots_i a\ldots),\Gamma\vdash\Delta}{t:\phi,F(\ldots_i t\ldots),\Gamma\vdash\Delta}\mathrm{Relpace}^a_t}{F(\ldots_i t\ldots),\Gamma\vdash\Delta}\mathrm{Cut}}{\Gamma\vdash\Delta}\mathrm{Cut}$$

where the $\mathrm{Replace}^a_t$ operation transforms the proof of $a:\phi, F(\ldots_i a\ldots),\Gamma\vdash\Delta$ into a proof of $t:\phi, F(\ldots_i t\ldots),\Gamma\vdash\Delta$ by replacing all occurrences of the parameter a by the term t. In the jargon, a is called an *eigenvalue* of the former proof.

In the calculi S_1 and S_2, the *Naming Rule* commutes with *Cut*, as it should. This is shown by the following transformation (together with a similar transformation for *Cut*s with a *Name* rule application on the right-hand branch):

$$\dfrac{\dfrac{s,\Gamma\vdash\Delta,\phi}{\Gamma\vdash\Delta,\phi}\mathrm{Name} \qquad \phi,\Gamma\vdash\Delta}{\Gamma\vdash\Delta}\mathrm{Cut} \qquad \leadsto \qquad \dfrac{\dfrac{s,\Gamma\vdash\Delta,\phi \qquad \dfrac{\phi,\Gamma\vdash\Delta}{s,\phi,\Gamma\vdash\Delta}\mathrm{Weak}}{s,\Gamma\vdash\Delta}\mathrm{Cut}}{\Gamma\vdash\Delta}\mathrm{Name}$$

The real problem arises when we come to look at the *Term* rule. It does not commute with the *Cut* rule. The obvious transformation

$$\dfrac{\dfrac{s,\Gamma\vdash\Delta,\phi}{\Gamma',\Gamma\vdash\Delta,\Delta',\phi}\mathrm{Term} \qquad \phi,\Gamma',\Gamma\vdash\Delta,\Delta'}{\Gamma',\Gamma\vdash\Delta,\Delta'}\mathrm{Cut} \qquad \leadsto \qquad \dfrac{\dfrac{\dfrac{s,\Gamma\vdash\Delta,\phi}{s,\Gamma',\Gamma\vdash\Delta,\Delta',\phi}\mathrm{Weak} \qquad \dfrac{\phi,\Gamma',\Gamma\vdash\Delta,\Delta'}{s,\phi,\Gamma',\Gamma\vdash\Delta,\Delta'}\mathrm{Weak}}{s,\Gamma',\Gamma\vdash\Delta,\Delta'}\mathrm{Cut}}{\Gamma',\Gamma\vdash\Delta,\Delta'}\mathrm{Term}$$

cannot be guarantied to yield a correct proof. The Γ' and Δ' may contain formulae which are not propositions and so the application of the *Term* rule in the transformed structure may not respect its side condition. An example of an *in*eliminable *Cut* arising from an application of the *Term* rule is given below:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Axiom}
\UnaryInfC{$s, \sigma \vdash s' : (s : \sigma), s$}
\AxiomC{}
\RightLabel{Axiom}
\UnaryInfC{$s, \sigma \vdash s' : (s : \sigma), \sigma$}
\RightLabel{R:}
\BinaryInfC{$s, \sigma \vdash s' : (s : \sigma), s : \sigma$}
\AxiomC{}
\RightLabel{Axiom}
\UnaryInfC{$s', s : \sigma \vdash s'$}
\AxiomC{}
\RightLabel{Axiom}
\UnaryInfC{$s', s : \sigma \vdash \sigma$}
\RightLabel{R:}
\BinaryInfC{$s', s : \sigma \vdash s' : (s : \sigma)$}
\RightLabel{Term}
\UnaryInfC{$s : \sigma, s, \sigma \vdash s' : (s : \sigma)$}
\RightLabel{Cut}
\BinaryInfC{$s, \sigma \vdash s' : (s : \sigma)$}
\end{prooftree}
$$

The only complex formula in the end-sequent if the formula $s' : (s : \sigma)$, so the final rule application can only be an instance of R:, *Term* or *Cut*. R: cannot be applied since the sequent $s, \sigma \vdash s'$ is clearly not valid and an application of the *Term* rule would be equally useless: since both s and σ are not propositions, the resulting sequent would be $s' \vdash s' : (s : \sigma)$ which is also not valid. The *only* possibility is an application of the *Cut* rule.

Our solution — inspired by the preceding discussion of *Parity* — is to extend the calculus by adding the *Upwards Parity* rule. In fact, we will add two new rules from which the UP rule can be derived. We could have added UP directly, but the resulting calculus would be less user-friendly. The two new rules are:

$$
\frac{a, a : \phi, \Gamma \vdash \Delta}{a, \phi, \Gamma \vdash \Delta}\text{LUP} \qquad\qquad \frac{a, \Gamma \vdash \Delta, a : \phi}{a, \Gamma \vdash \Delta, \phi}\text{RUP}
$$

where a is a *new* parameter (or just a new situation name when we restrict to L_0), i.e., one that does not occur in ϕ or in any of the formulae in Γ or Δ. Together with the *Naming Rule* (which we will now need to add even in the non-quantificational calculus) the rules LUP and RUP are sufficient for the cut-free derivability of *Upwards Parity*. Both rules can be seen to be correct with respect to the semantics of all our languages (the proof is similar to that of Theorem 3).

Theorem 12 Cut Elimination. Any proof in S_2 can be transformed into a *Cut*-free proof in S_2+LUP+RUP with the same end-sequent. [6]

PROOF: *(Outline.)* The details of any (syntactic) cut-elimination theorem are far to messy to display in a paper of this length, so we will be content with a rough outline. The basic strategy is to stay close to Gentzen's *Hauptsatz*. Many of our rules are inherited directly from his formulation of a sequent calculus for elementary classical logic, and our languages are sufficiently similar to the languages of elementary logic that something like Gentzen's complexity rank for *Cut*'s will suffice.[7]

[6] Similar results can be obtained for the weaker calculi S_0 and S_1, remembering that we have to add the *Name* rule to S_0 first.

[7] Care must be taken in defining the complexity of generalized terms: elementary terms still have rank 0, but the generalized term $F(\phi_1, \ldots, \phi_n)$ (where at least one of the ϕ's is not a term) has rank given by the maximum of the ranks of $\phi_1, \ldots \phi_n$ plus 2.

Suppose we have an S_2 proof π of the sequent $\Gamma_0 \vdash \Delta_0$. We will assume that every application of the *Term* rule occurs immediately below an application of L: or R:. Equivalently, we could work in the calculus $S_2 - \mathit{Term}$+TL: +TR:. We transform π in four stages.

1. Transform π to π' by eliminating all applications of the *Cut* rule using the standard Gentzen transformations for each of the classical rules together with the transformations outlined in the preceding discussion. Note that π', though *Cut*-free, is not necessarily an S_2 proof since it may contain inadmissible applications of the *Term* rule.
2. Replace every non-propositional formula ϕ occurring in π', *except* terms introduced by applications of the *Term* rule, by the formula $a : \phi$, where a is a parameter which does not occur in any formula of π'. Call the resulting structure π''. Note that applications of the *Term* rule are now acceptable, but that applications of virtually every other rule are not!
3. Expand each rule application into a *Cut*-free S_2+LUP+RUP proof of its conclusion from its premises. Call the resulting proof π'''. We should examine each case, but here we will be content with an example. An application of the L1$\wedge$ rule in the *Cut*-free structure π' is transformed (at stage 2.) by prefixing non-propositions with 'a :' and expanded (at stage 3.) into a *Cut*-free proof in the calculus S_2+LUP+RUP

$$\dfrac{s, \phi, \Gamma', \Gamma \vdash \Delta, \Delta'}{s, \phi \wedge \phi', \Gamma', \Gamma \vdash \Delta, \Delta'}\,\mathrm{L1}\wedge \quad \overset{2.}{\leadsto} \quad \dfrac{s, a : \phi, a : \Gamma', \Gamma \vdash \Delta, a : \Delta'}{s, a : (\phi \wedge \phi'), a : \Gamma', \Gamma \vdash \Delta, a : \Delta'} \quad \overset{3.}{\leadsto}$$

$$\dfrac{\dfrac{\dfrac{}{a, a : \Gamma', \Gamma \vdash \Delta, a : \Delta', a}\,\mathrm{Axiom} \qquad \dfrac{\dfrac{\dfrac{s, a : \phi, a : \Gamma', \Gamma \vdash \Delta, a : \Delta'}{a, a : \phi, a : \Gamma', \Gamma \vdash \Delta, a : \Delta'}\,\mathrm{Term}}{a, \phi, a : \Gamma', \Gamma \vdash \Delta, a : \Delta'}\,\mathrm{LUP}}{a, \phi \wedge \phi', a : \Gamma', \Gamma \vdash \Delta, a : \Delta'}\,\mathrm{L1}\wedge}{a, a : (\phi \wedge \phi'), a : \Gamma', \Gamma \vdash \Delta, a : \Delta'}\,\mathrm{L:}}{s, a : (\phi \wedge \phi'), a : \Gamma', \Gamma \vdash \Delta, a : \Delta'}\,\mathrm{Term}$$

   where Γ and Δ are sets of propositions, Γ' and Δ' are sets of non-propositions and s is a term introduced by an application of the *Term* rule (which may or may not occur in this part of π', but which is included for the sake of generality).

3. Finally, extend the proof π''' — which will have an end-sequent of the form $a : \Gamma', \Gamma \vdash \Delta, a : \Delta'$ where Γ and Δ are propositions, Γ' and Δ' are non-propositions, $\Gamma \cup \Gamma' = \Gamma_0$ and $\Delta \cup \Delta' = \Delta_0$ — to a proof of the sequent $\Gamma_0 \vdash \Delta_0$. This can be done by *Weakening* π''' to give a proof with end-sequent $a, a : \Gamma', \Gamma \vdash \Delta, a : \Delta'$, applying the rules LUP and RUP as many times as is necessary to obtain a proof of $a, \Gamma', \Gamma \vdash \Delta, \Delta'$ and then — with a final application of the *Name* rule — obtaining a proof of $\Gamma', \Gamma \vdash \Delta, \Delta'$, otherwise known as $\Gamma_0 \vdash \Delta_0$.

$\square$

Define S_2^* to be the calculus obtained by dropping the *Cut* and *Term* rules from S_2 and adding the rules TL:, TR:, LUP and RUP. As an example of an S_2^* proof consider the previously troublesome derivation of the sequent $s, \sigma \vdash s' : (s : \sigma)$ with its *Cut* removed:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Axiom}
\UnaryInfC{$a, s \vdash s : \sigma, a$}
\AxiomC{}
\RightLabel{Axiom}
\UnaryInfC{$a, s, \sigma \vdash s$}
\AxiomC{}
\RightLabel{Axiom}
\UnaryInfC{$a, s, \sigma \vdash \sigma$}
\RightLabel{R:}
\BinaryInfC{$a, s, \sigma \vdash s : \sigma$}
\RightLabel{L:}
\BinaryInfC{$a, s, a : \sigma \vdash s : \sigma$}
\RightLabel{TL:}
\UnaryInfC{$s', a : s, a : \sigma \vdash s : \sigma$}
\RightLabel{TR:}
\UnaryInfC{$a, a : s, a : \sigma \vdash s' : (s : \sigma)$}
\RightLabel{LUP}
\UnaryInfC{$a, a : s, \sigma \vdash s' : (s : \sigma)$}
\RightLabel{LUP}
\UnaryInfC{$a, s, \sigma \vdash s' : (s : \sigma)$}
\RightLabel{Name}
\UnaryInfC{$s, \sigma \vdash s' : (s : \sigma)$}
\end{prooftree}
$$

Corollary 13 Correctness and Completeness. A sequent of L_2 formulae is provable in the calculus S_2^* iff it is L_2 valid.

PROOF: From the *Cut* elimination theorem together with the correctness and completeness of S_2 as proved in Theorem 9. □

Corollary 14 ***Semi*-Analyticity.** The only formulae occurring in an S_2^* proof are of the form a, $a : \phi$ or ϕ where ϕ is a subformula of a formula in the end-sequent and a is a parameter.

PROOF: All the rules of S_2^* obey the condition and so S_2^* proofs do too. □

We think these result can be improved, conjecturing that S_2^* proofs can be transformed into proofs in which the only applications of the rules LUP and RUP occur in proving the end-sequent from a sequent of the form $a : \Gamma', \Gamma \vdash \Delta, a : \Delta'$ where Γ, Δ and Γ', Δ' are the propositional and non-propositional parts of the end-sequent, respectively, and a is a new parameter. If this is right then we have the stronger result that

1. S_2^* with the rules restricted in this way is correct and complete, and
2. every formula occurring in an S_2^* proof in which the rules are so restricted is a subformula of the sequent $a : \Gamma', \Gamma \vdash \Delta, a : \Delta'$.

Such results should lead to efficient methods for automatic theorem proving in each of our calculi (following Wallen 1989).

6 Conservativity and Change

In this section we will indicate some of the connections between the present calculi and other logical systems and suggest further lines of research.

The languages L_0, L_1 and L_2 are strictly contained in each other in an increasing sequence.[8] So too are the sublanguages of their terms, which we will denote by T_0, T_1 and T_2. But there are some more interesting sublanguages which we will now examine.

Elementary Logic. The language L_e of elementary (i.e., first-order) logic with the function symbols and relation symbols of L_2 is a strict sublanguage of L_2. T_2 consists of precisely the elementary terms. If we interpret the equality predicate as ':' then the language $L_{e=}$ of elementary logic with equality is also a sublanguage of L_2. When restricted to these languages the calculus S_2 (and S_2^*) licenses all the standard sequent rules for classical elementary logic and classical elementary logic with equality. By the *Cut* elimination theorem we can see that S_2 is therefore a conservative extension of these logics.

Conversely, we can provide a translation of L_2 into $L_{e=}^+$ which shows that no additional expressive power has been gained; we are still first-order. $L_{e=}^+$ is just like $L_{e=}$ but with relation symbols increased in arity by 1. We define a translation $\{t\} : L_2 \to L_{e=}^+$ for each term t which is straightforwardly inductive but with base cases

1. $\{t\}f(x_1, \ldots, x_n) \mapsto t = f(x_1, \ldots, x_n)$
2. $\{t\}R(x_1, \ldots, x_n) \mapsto R(t, x_1, \ldots, x_n)$
3. $\{t\}x_1 : x_2 \mapsto x_1 = x_2$
4. $\{t\}F(\phi_1, \ldots, \phi_n) \mapsto \exists x_1 \ldots x_n(\{x_1\}\phi_1 \wedge \ldots \{x_n\}\phi_n \wedge F(x_1 \ldots x_n))$

The translation given by any term t is truth preserving in that a pseudo-formula ϕ of L_2 is true at a situation i (w.r.t. α, g) iff the formula $\{t\}\phi$ of $L_{e=}^+$ is true in the elementary model which is "generated" by i (w.r.t. g).

Modal Logic. Consider the L_2 language with no function symbols, one unary relation symbol $\Diamond$ and a countably infinite set of 0-ary relation symbols (or "infon symbols"). Define $\Box\phi$ to be $\neg\Diamond\neg\phi$ and restrict to the non-parametric non-propositional quantifier-free fragment of the language. The result is the language $L_\Box$ of sentential modal logic. Moreover, the axioms and rules of the modal logic K are derivable in S_2. Again, we have a conservativity result: only valid sequents of K are derivable in S_2 restricted to $L_\Box$.

We could extend $L_\Box$ with more unary relation symbols (i.e., with more modal operators) to get poly-modal versions of K and we could add elementary atomic formulae and quantification to get first-order K with a constant domain interpretation. The Barcan formula is provable.

[8] We interpret the "situation names" and "infon symbols" of L_0 and L_1 as 0-ary function and relation symbols, respectively, in L_2.

The conservativity results give a potential proof-theoretic insight into frame-completeness theorems in modal logic. Given an elementary frame condition expressed by the $L_{e=}$ formula C and a (normal) modal logic with axioms M, the completeness of M with respect to C frames is equivalent to the admissibility in S_2 of the rule:

$$\frac{C, M, \Gamma \vdash \Delta}{M, \Gamma \vdash \Delta}$$

Blackburn's (1990) Nominal Modal Logic is obtained by adding a countably infinite set of 0-ary function symbols — called *nominals* — to $L_\Box$. This can be extended to Gargov & Goranko's (1991) language by adding an $S5$ "possibility" operator M, which can be defined by $M\phi = \phi : \phi$. Bull's (1970) language is obtained by adding the first-order quantifiers. Reape's (1990) languages for feature-value logics — used to describe linguistic formalisms — are also sub-languages of L_2, but we require extra axioms to disallow "constant clashes" ($\neg(c_1 : c_2)$) and for the feature labels, which are unary relation symbols l such that $\forall x_1 x_2 (l x_1 \wedge l x_2 \rightarrow x_1 : x_2)$.

There are many other connections to modal logic on the proof-theoretic side. S_0 seems to share many features with sequent calculi for modal $S5$. See, for example, Fitting (1983). Also there are strong resemblances to the topological logic of Rescher and Garson (1968).

Intuitionistic and Partial Logic. The issue of what kind of negation to have in a logic of situations has always been far from clear. Probably, it would be wise to consider a number of different negations. In focusing on classical negation in this paper we have certainly not done justice to the traditional thinking of situation theorists. Nevertheless, there is something to be said for starting with the simplest! We hope to provide a version of the calculi with intuitionistic parity by restricting sequents to having one formula on the right-hand side. Likewise, three-valued approaches should be relatively easy to incorporate.

Perspectives. The ability of these languages to say that a situation supports information of the form $s : \phi$ allows a partial formalization of the notion of a *perspective* in situation theory (Seligman 1989,1990a). In L_0, for example, relative to a theory Γ, each term t defines a classification C_t of situation (terms) by formulae: s is classified in C_t as being of type σ iff the sequent $\Gamma \vdash t : (s : \phi)$ is provable. Constraints between formulae could be defined either via entailment relative to Γ giving a "logical perspective" in the sense of Seligman (1989), or — more interestingly — via provable functional or relational connections between terms (the "information channels" of Seligman (1990b)). However, there is one major limitation of this approach: the *Parity Principle* ensure that every situation will generate the same classification!

In first-order modal logic where there is a distinction between models with a constant-domain, cumulative-domains or variable-domains of quantification at each world. The calculi we have developed here are analogous to the constant-domain modal logics. We need to explore the possibility of cumulative and variable domains within the context of situation theory. There is, of course, a precedent for the cumulative-domain approach in Barwise's writings about the 'part-of' relation between situations (Barwise 1989). Pursuing the connection with first-order modal logic further, we note that all our terms are acting like *rigid designators*. This is another assumption which needs to be questioned: perhaps a systematic variation of reference across situations — such as is studied in Rescher and Garson (1968) — would be more rewarding. The proof-theoretic consequences of these semantic changes are quite unknown.

Higher-Order Intensional Situated Logic There are, of course, many issues associated with situation theory which this paper does not address. The distinction between infons and propositions, the fine-grained intensionality of situation theoretic objects, higher-order quantification and abstraction are all omitted. Nevertheless, as we indicated in our introduction, we hope that "standard" techniques will be of help here. In particular, the illative logic of Plotkin (1989) has already bridged the gap between the more usual type-free intensional logics and situation theory. We hope to combine insights from that paper and the present one by "situating" Plotkin's logic.

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