Hybrid logics with Sahlqvist axioms

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Abstract

We show that every extension of the basic hybrid logic with modal Sahlqvist axioms is complete. As a corollary of our approach, we also obtain the Beth property for a large class of hybrid logics. Finally, we show that the new completeness result cannot be combined with the existing general completeness result for pure axioms.

1 Introduction

Hybrid logic comes with a general completeness result: Every extension with pure axioms of the basic hybrid logic with [name] and [bg] rules is complete [2, 3]. These rules are recalled below. A pure axiom is a formula constructed from nominals only, thus not containing arbitrary proposition letters. Pure axioms correspond to first order frame conditions and are quite expressive [2]. For instance, $i \to \neg\Diamond i$ defines the class of irreflexive frames.

We can compare this general result with Sahlqvist's theorem for modal logic, a similar general completeness result. Several questions come to mind. Is every modal Sahlqvist axiom expressible as a pure axiom? No, the Church–Rosser axiom $\Diamond\Box p \to \Box\Diamond p$ is a counterexample [6]. This gives us two new questions:

1. Is the extension of the basic hybrid logic with a set of modal Sahlqvist axioms always complete? That is, does Sahlqvist's theorem go through for hybrid logic?
2. Can we combine the two general completeness results? That is, is every extension of the basic hybrid logic with a set Σ of modal Sahlqvist axioms and a set Π of pure axioms complete with respect to the class of frames defined by Σ and Π together?

This paper answers both questions. We show that every extension of the basic hybrid logic with modal Sahlqvist axioms is complete even without the [name] and [bg] rules. The second question on the other hand must be answered in the negative: We give a modal Sahlqvist formula σ and a pure formula π such that the basic hybrid logic extended with the axioms σ and π is incomplete even in the presence of the [name] and the [bg] rules.

As a corollary of the Sahlqvist theorem for hybrid logic, we solve an open problem in hybrid logic: whether Beth's definability property holds (cf. [5] for a discussion of this open problem and some partial results). Another corollary of our analysis is that [name] and [bg] are superfluous, not only for the basic hybrid system, but also for every extension with Sahlqvist axioms. This is a desirable result, since these rules are non-orthodox in the sense that they involve syntactic side-conditions, much like Gabbay's irreflexivity rule.

The paper is organized as follows. Section 2 briefly recalls hybrid logic. Section 3 shows Sahlqvist's theorem for hybrid logic. In Section 4 we derive interpolation and Beth's property. Section 5 shows that a combination of Sahlqvist and pure axioms is not guaranteed to be complete. We conclude in Section 6.

2 Hybrid logic

What follows is a short textbook-style presentation of hybrid logic, following [2]. Hybrid logic is the result of extending the basic modal language with a second sort of atomic propositions called nominals, and with satisfaction operators. Nominals are proposition letters whose interpretation in models is restricted to singleton sets. In other words, nominals act as names for worlds. Satisfaction operators allow one to express that a formula holds at the world named by nominal. For example, $@_i p$ expresses that p holds at the world named by the nominal i.

Formally, let PROP be a countably infinite set of proposition letters and NOM a countably infinite set of nominals (the results discussed in this paper and their proofs also apply if NOM is finite). The formulas of the basic hybrid logic are given as follows.

$$\phi ::= p \mid i \mid \top \mid \neg\phi \mid \phi \wedge \psi \mid \Box\phi \mid @_i\phi$$

where $p \in$ PROP and $i \in$ NOM. Models are of the form $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is a frame and V a valuation function for the proposition letters and nominals. The truth definition for the nominals is the same as for the proposition letters: $\mathfrak{M}, w \models i$ iff $w \in V(i)$. The only difference is in the admissible valuations: only valuation functions are allowed that assign to each nominal a singleton set. The interpretation of the satisfaction operators is as could be expected: $\mathfrak{M}, w \models @_i\phi$ iff $\mathfrak{M}, v \models \phi$, where $V(i) = \{v\}$.

Next, let us turn to axiomatizations for this language. Let Δ be the set of axioms given by [agree], [propagation], [elimination], [ref], and [self-dual], for all $i, j \in$ NOM.

[agree]	$@_i p \to @_j @_i p$
[propagation]	$@_i p \to \Box @_i p$
[elimination]	$(@_i p \wedge i) \to p$
[ref]	$@_i i$
[self-dual]	$@_i p \leftrightarrow \neg @_i \neg p$.

Let $\mathbf{K}_{\mathcal{H}(@)}$ be the smallest set of formulas containing all propositional tautologies, the axioms $\Box(p \to q) \to (\Box p \to \Box q)$ and $@_i(p \to q) \to (@_i p \to @_i q)$ for $i \in$ NOM, and the axioms in Δ, closed under *modus ponens*, *uniform substitution of formulas for proposition letters and nominals for nominals*, *generalization* (If $\vdash \phi$ then $\vdash \Box\phi$), and *@-generalization* (If $\vdash \phi$ then $\vdash @_i\phi$). Given a set Σ of hybrid formulas, the logic $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ is obtained by adding the formulas in Σ to $\mathbf{K}_{\mathcal{H}(@)}$ as extra axioms, and closing under the given rules.

Theorem 2.1 (Basic completeness)
$\mathbf{K}_{\mathcal{H}(@)}$ is sound and strongly complete for the class of all frames.

A sketch of the proof of Theorem 2.1 can be found in [2]. Theorem 2.1 also follows from Theorem 3.4 below.

A general completeness result holds for extensions of the basic logic with pure axioms, provided two extra derivation rules are added to the calculus. A pure axiom is an axiom that contains no proposition letters, such as $\Diamond\Diamond i \to \Diamond i$ (which defines transitivity) and $i \to \neg\Diamond i$ (which defines irreflexivity). The relevant derivation rules are the following:

[name] If $\vdash @_i\phi$ and i does not occur in ϕ, then $\vdash \phi$
[bg] If $\vdash @_i\Diamond j \to @_j\phi$ where $j \neq i$ and j does not occur in ϕ, then $\vdash @_i\Box\phi$

Several variants of these rules occur in literature, under names such as *Cov* [6] and *Name and Paste* [2]. It the above shape, the rules first appear in [3].

Let $\mathbf{K}^+_{\mathcal{H}(@)}$ be the logic obtained by adding these two derivation rules to $\mathbf{K}_{\mathcal{H}(@)}$. Furthermore, given a set Σ of hybrid formulas, the logic $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ is obtained by adding the formulas in Σ to $\mathbf{K}^+_{\mathcal{H}(@)}$ as extra axioms, and closing under all rules, including the two extra rules.

Theorem 2.2 (Pure completeness [3])
Let Σ be any set of pure formulas. Then $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ is sound and strongly complete for the frame class defined by Σ.

This paper is about strengthening these two theorems. In the next section we extend Theorem 2.1 to hybrid logics axiomatized by modal Sahlqvist formulas. In section 5 it is shown that it is not possible to combine this Sahlqvist completeness theorem with Theorem 2.2.

3 Sahlqvist completeness for hybrid logic

The definition of Sahlqvist formulas is unfortunately not as simple as that of pure formulas. We will use the definition from [2], which applies to modal formulas containing several, not necessarily unary, modalities. A modal formula is said to be *positive* (*negative*) if all occurrences of proposition letters in the formula are in the scope of an even (odd) number of negations. A *boxed atom* is a formula of the form $\Box_1 \cdots \Box_n p$, where $\Box_1, \ldots, \Box_n$ are unary box-modalities ($n \geq 0$) and p is a proposition letter. A *modal Sahlqvist formula* is a modal formula of the form $\phi \to \psi$, where ψ is positive and ϕ is built up from $\top$, $\bot$, boxed atoms and negative formulas using $\wedge$, $\vee$ and diamond-modalities of any arity. Note that nullary modalities (also known as modal constants) are freely allowed in ϕ. For instance, for a modal constant δ, $\delta \to \neg\Diamond\delta$ is a Sahlqvist formula, since both δ and $\neg\Diamond\delta$ are positive as well as negative formulas.

Theorem 3.1 (Sahlqvist completeness for modal logic [11, 2])
Every extension of the basic modal logic with a set Σ of modal Sahlqvist axioms is complete with respect to the frame class defined by Σ. This holds also for languages containing several, not necessarily unary, modalities.

It should be mentioned that this result is not optimal. For instance, Goranko and Vakarelov [8] have recently proposed a slightly larger class of modal formulas for which Theorem 3.1 holds.

Our aim is to prove a similar theorem for hybrid logic. To this end, we will view the hybrid language as an ordinary modal language, in which each satisfaction operator $@_i$ is a separate unary (box) modality, and each nominal is a nullary modality. Under this non-standard interpretation of the hybrid language, the axioms [agree], [propagation], [elimination], [ref] and [self-dual] are modal axioms expressing certain frame conditions. The frames are of the form $\mathfrak{F} = \langle W, R, (R_i)_{i \in nom}, (S_i)_{i \in \text{NOM}} \rangle$, where each R_i is a binary relation on W and each S_i is a subset of W. We will call such frames *non-standard frames*, to distinguish them from the ordinary frames. A non-standard model is a pair $(\mathfrak{F}, V)$ where $\mathfrak{F}$ is a non-standard frame and the valuation V interprets the proposition letters (the interpretation of the nominals is already given by $\mathfrak{F}$). The interpretation of the nominals and satisfaction operators on such models is as expected: we simply extend the usual satisfaction definition for modal logic with the following clauses.

$$\begin{array}{lll} \mathfrak{M}, w \models i & \text{iff} & w \in S_i, \\ \mathfrak{M}, w \models @_i\phi & \text{iff} & \forall w'(wR_iw' \Longrightarrow \mathfrak{M}, w' \models \phi). \end{array}$$

Let us say that such a non-standard frame or model is *nice* if for each $i \in \text{NOM}$, S_i is a singleton and $\forall xy(R_ixy \leftrightarrow S_iy)$. There is a natural correspondence between nice non-standard models and standard hybrid models: with each nice non-standard model $\mathfrak{M} = \langle W, R, (R_i)_{i \in \text{NOM}}, (S_i)_{i \in \text{NOM}}, V \rangle$, we can associate a standard hybrid model $\mathfrak{M}^+ = \langle W, R, V \cup \{(i, S_i) \mid i \in \text{NOM}\} \rangle$. This operation on models is bijective, in the sense that for every standard hybrid model $\mathfrak{M}$, there is exactly one nice non-standard model $\mathfrak{N}$ such that $\mathfrak{M} = \mathfrak{N}^+$. Moreover, $(\cdot)^+$ preserves local truth of formulas, in the following sense.

Lemma 3.2
For all nice non-standard models $\mathfrak{M}$, worlds w and hybrid formulas ϕ, $\mathfrak{M}, w \models \phi$ iff $\mathfrak{M}^+, w \models \phi$. Moreover, if ϕ contains no nominals or satisfaction operators, then ϕ is valid on the underlying frame of $\mathfrak{M}^+$ iff ϕ is valid on the underlying (non-standard) frame of $\mathfrak{M}$.

Proof. A straightforward inductive argument. ∎

As we mentioned already, the axioms in Δ define properties of non-standard frames. For instance, [self-dual] says that the relations R_i are functional. As a matter of fact, each of the axioms is either a Sahlqvist formula, or is equivalent to a conjunction of Sahlqvist formulas ([ref] is equivalent to $\top \to @_i i$ and [self-dual] is equivalent to $(@_i p \to \neg @_i \neg p) \wedge (\neg @_i \neg p \to @_i p)$). Hence, each has a first-order correspondent, given below.

$$\begin{array}{rl} \text{[agree]} & \forall xyz(R_jxy \wedge R_iyz \to R_ixz) \\ \text{[propagation]} & \forall xyz(Rxy \wedge R_iyz \to R_ixz) \\ \text{[elimination]} & \forall x(S_ix \to R_ixx) \\ \text{[ref]} & \forall xy(R_ixy \to S_iy) \\ \text{[self-dual]} & \forall x \exists! y(R_ixy). \end{array}$$

Lemma 3.3
A point-generated non-standard frame $\mathfrak{F}$ is *nice* iff $\mathfrak{F} \models \Delta$.

Proof. The left-to-right direction is obvious. For the right-to-left direction, suppose $\mathfrak{F}$ is a non-standard frame generated by a world w, and $\mathfrak{F} \models \Delta$. By [ref] and [self-dual], $|S_i| \geq 1$ for each $i \in \textsc{nom}$. In order to see that $|S_i| \leq 1$, let $v, u \in S_i$ for some $i \in \textsc{nom}$. By [elimination], vR_iv and uR_iu, and by point-generatedness, v and u are reachable from w in finitely many steps along the accessibility relations. It follows by repeated application of [agree] and [propagation] that wR_iv and wR_iu. Hence, by [self-dual], $v = u$. Thus, we have shown that $|S_i| = 1$ for all $i \in \textsc{nom}$. Finally, it follows by [self-dual] and [ref] that $\forall xy.(R_ixy \leftrightarrow S_iy)$ for all $i \in \textsc{nom}$. In other words, $\mathfrak{F}$ is nice. ■

For any set Σ of hybrid formulas, let $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ be the logic obtained by adding the formulas in Σ to $\mathbf{K}_{\mathcal{H}(@)}$ as axioms and closing under the derivation rules of $\mathbf{K}_{\mathcal{H}(@)}$.

Theorem 3.4 (Sahlqvist completeness for hybrid logic)
Let Σ be a set of modal Sahlqvist formulas not containing nominals or satisfaction operators. Then $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ is sound and strongly complete for the class of frames defined by Σ.

Proof. Soundness is obvious. In what follows, we will prove strong completeness. Let Σ be a set of modal Sahlqvist formulas. It follows by Theorem 3.1 that $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ is sound and strongly complete with respect to the class of non-standard frames validating $\Delta\cup\Sigma$ (here we use our earlier observation that each axiom in Δ is equivalent to a Sahlqvist formula).

Now, consider any $\mathbf{K}_{\mathcal{H}(@)}\Sigma$-consistent set of hybrid formulas Γ. Then there is a non-standard model $\mathfrak{M}$ and a world w such that $\mathfrak{M}, w \models \Gamma$ and the underlying (non-standard) frame of $\mathfrak{M}$ validates $\Sigma \cup \Delta$. We may assume without loss of generality that $\mathfrak{M}$ is point generated from w. Then, by Lemma 3.3, $\mathfrak{M}$ is nice. Now, consider the standard hybrid model $\mathfrak{M}^+$. By Lemma 3.2, $\mathfrak{M}^+, w \models \Gamma$ and the underlying (standard) frame of $\mathfrak{M}^+$ validates Σ (recall that the formulas in Σ do not contain any nominals or satisfaction operators). In other words, Γ is satisfiable on a Σ-frame. ■

Gargov and Goranko [6] obtain a similar result for the hybrid language with the global modality, via a slightly different route. Note that Theorem 3.4 does not make use of the [name] and [bg] derivation rules. These rules are admissible in extensions of the basic hybrid logic with modal Sahlqvist formulas.

4 Interpolation and Beth's property

It was an open problem whether the basic hybrid logic has *Beth's definability property* [5]. Intuitively, Beth's property is a completeness theorem for definitions: it states that every implicit, semantic definition corresponds to an explicit, syntactic, definition. More concretely, we say that a set of formulas Γ *implicitly defines* a proposition letter p in a modal or hybrid logic Λ if whenever a Λ-model $\mathfrak{M}$ globally satisfies Γ as well as $\Gamma[p/p']$, then p and p' have the same valuation in $\mathfrak{M}$.[1] The Beth definability

[1]Here, by a Λ-model we mean a model based on frame for Λ, p' is a proposition letter not occurring in Γ, and $\Gamma[p/p']$ is the result of replacing all occurrences of p by p' in Γ.

property states that whenever this obtains, there exists a formula ϑ in which p does not occur, such that every Λ-model $\mathfrak{M}$ globally satisfying Γ globally satisfies $p \leftrightarrow \vartheta$. The relevant formula ϑ is called an *explicit definition* of p relative to Γ and Λ.

Usually, the Beth property is proved as a corollary of the interpolation property. Unfortunately, it is known that interpolation fails for hybrid logic [1]. In what follows, we will show that the basic hybrid logic satisfies a restricted version of the interpolation, that is strong enough to entail the Beth property. In fact, this restricted version of interpolation holds not only for the basic hybrid logic, but for many other hybrid logics as well.

Let us say that a modal or hybrid logic Λ has *interpolation over proposition letters* if whenever $\phi \to \psi \in \Lambda$, there exists a formula ϑ, such that $\phi \to \vartheta \in \Lambda$ and $\vartheta \to \psi \in \Lambda$, and all proposition letters (but not necessarily the nominals) occurring in ϑ occur both in ϕ and in ψ.

THEOREM 4.1 (Interpolation for hybrid logic)
Let Σ be a set of modal Sahlqvist formulas. If all axioms in Σ have universal Horn frame correspondents, then $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ has interpolation over proposition letters.

PROOF. We again apply the non-standard modal semantics of the hybrid language, and use the fact that every modal logic axiomatized by Sahlqvist formulas with universal Horn correspondents has interpolation over proposition letters [9, Corollary B.4.1]. By assumption, all axioms in Σ are modal Sahlqvist formulas with universal Horn correspondents. Furthermore, with the exception of [self-dual], all first-order correspondents of axioms in Δ are universal Horn sentences. The [self-dual] itself is equivalent to the conjunction of $\neg @_i \neg p \to @_i p$ and $\neg @_i \neg \top$. The former is a modal Sahlqvist formula with a universal Horn correspondent and the latter is a formula without proposition letters. Rautenberg [10] proved that extending a modal logic that has interpolation over proposition letters with formulas without proposition letters yields again a logic with interpolation over proposition letters. ∎

COROLLARY 4.2 (The Beth property for hybrid logic)
Let Σ be a set of modal Sahlqvist formulas not containing nominals or satisfaction operators. If all axioms in Σ have universal Horn frame correspondents, then $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ has Beth's definability property.

PROOF. Let Γ be any set of hybrid formulas, and p a proposition letter, such that Γ implicitly defines p in $\mathbf{K}_{\mathcal{H}(@)}\Sigma$. Let p' be a new proposition letter, and let $\Gamma[p/p']$ be the result of replacing all occurrences of p in Γ by p'. Let K_Σ be the frame class defined by Σ. Then by definition, every model based on a frame in K_Σ that globally satisfies $\Gamma \cup \Gamma[p/p']$ globally satisfies $p \leftrightarrow p'$. Now, let $\Gamma^\Box$ be the set of formulas

$$\{\Box^n \phi, @_i \Box^n \phi \mid \phi \in \Gamma, n \in \omega, i \in \textsc{nom}\}$$

and let $\Gamma^\Box[p/p']$ be obtained from $\Gamma^\Box$ by replacing all occurrences of p by p'.

Claim: For each model $\mathfrak{M}$ based on a frame in K_Σ, and for each world w of $\mathfrak{M}$, if $\mathfrak{M}, w \models \Gamma^\Box \cup \Gamma^\Box[p/p']$ then $\mathfrak{M}, w \models p \leftrightarrow p'$.

Proof of claim: Suppose $\mathfrak{M}, w \models \Gamma^\Box \cup \Gamma^\Box[p/p']$. Let $\mathfrak{M}_w$ be the submodel of $\mathfrak{M}$ generated by w. By closure under generated subframes, the underlying frame of $\mathfrak{M}_w$

is in K_Σ. Moreover, by construction of $\Gamma^\Box$ and $\Gamma^\Box[p/p']$, $\mathfrak{M}_w$ globally satisfies Γ and $\Gamma[p/p']$). It follows that $\mathfrak{M}_w, w \models p \leftrightarrow p'$ and hence $\mathfrak{M}, w \models p \leftrightarrow p'$. ⊣

Since Σ consists of modal Sahlqvist formulas, K_Σ is a Δ-elementary frame class (i.e., it is defined by a set of first-order formulas). We may therefore apply compactness and conclude from the claim that there are $\phi_1, \ldots, \phi_n \in \Gamma^\Box$ such that

$$K_\Sigma \models (\bigwedge_{k\leq n} \phi_k) \wedge (\bigwedge_{k\leq n} \phi_k[p/p']) \quad \rightarrow \quad (p \leftrightarrow p')$$

and hence

$$K_\Sigma \models \big(p \wedge \bigwedge_{k\leq n} \phi_k\big) \quad \rightarrow \quad \big((\bigwedge_{k\leq n} \phi_k[p/p']) \rightarrow p'\big)$$

By Theorem 3.4 $K_\Sigma \models \phi$ is equivalent to saying that $\phi \in \mathbf{K}_{\mathcal{H}(@)}\Sigma$. Thus by the interpolation theorem there is a formula ϑ such that

1. The proposition letters p and p' do not occur in ϑ.
2. $K_\Sigma \models (p \wedge \bigwedge_{k\leq n} \phi_k) \rightarrow \vartheta$.
3. $K_\Sigma \models \vartheta \rightarrow ((\bigwedge_{k\leq n} \phi_k[p/p']) \rightarrow p')$

It follows from 2. and 3. by uniform substitution that $K_\Sigma \models (\bigwedge_{k\leq n} \phi_k) \rightarrow (p \leftrightarrow \vartheta)$. Hence, whenever a model based on a frame in K_Σ globally satisfies Γ, it globally satisfies $p \leftrightarrow \vartheta$. ∎

5 Combining pure and Sahlqvist axioms

As we mentioned in the introduction, not every Sahlqvist axiom corresponds to a pure axiom. It is natural to ask if completeness obtains when we extend the basic hybrid logic $\mathbf{K}^+_{\mathcal{H}(@)}$ with a combination of pure and Sahlqvist axioms. The answer is negative.

THEOREM 5.1
There is a pure formula π and a modal Sahlqvist formula σ such that the hybrid logic $\mathbf{K}^+_{\mathcal{H}(@)}\{\pi, \sigma\}$ is not complete for the frame class defined by $\pi \wedge \sigma$.

PROOF. Consider the following axioms (the first-order frame conditions they define are given as well):

[cr]	$\Diamond\Box p \rightarrow \Box\Diamond p$	$\forall xyz(Rxy \wedge Rxz \rightarrow \exists u(Ryu \wedge Rzu))$
[nogrid]	$\Diamond(i \wedge \Diamond j) \rightarrow \Box(\Diamond j \rightarrow i)$	$\forall xyzu(Rxy \wedge Rxz \wedge Ryu \wedge Rzu \rightarrow y = z)$
[func]	$\Diamond p \rightarrow \Box p$	$\forall xyz(Rxy \wedge Rxz \rightarrow y = z)$

[cr] and [func] are Sahlqvist formulas and [nogrid] is pure. As can be easily seen from the first-order correspondents, every frame validating [cr] and [nogrid] validates [func]. However, we will show that [func] is not derivable in $\mathbf{K}^+_{\mathcal{H}(@)}\{[\text{cr}], [\text{nogrid}]\}$.

Consider ω^ω, i.e., the countably branching tree of infinite depth. Let $\mathfrak{F}$ be the general frame whose domain is ω^ω, whose accessibility relation is the child relation, and in which the admissible sets are exactly the finite and co-finite sets. It is not hard to see that this is indeed a general frame. We will claim that every axiom of

$\mathbf{K}^+_{\mathcal{H}(@)}\{[\text{cr}], [\text{nogrid}]\}$ is valid on $\mathfrak{F}$, and that all its derivation rules preserve validity on $\mathfrak{F}$. Since, $\mathfrak{F} \not\models [\text{func}]$, it follows that $[\text{func}] \notin \mathbf{K}^+_{\mathcal{H}(@)}\{[\text{cr}], [\text{nogrid}]\}$.

The only non-trivial part of our claim concerns the axiom [cr] and the derivation rules [name] and [paste]. That [name] and [paste] preserve validity on $\mathfrak{F}$ can be seen quite easily, using the fact that all singleton sets are admissible. Finally, to show that $\mathfrak{F} \models [\text{cr}]$, suppose $\mathfrak{F}, V, w \models \Diamond\Box p$. Since $V(p)$ admissible, it must be either finite or co-finite. Since w satisfies $\Diamond\Box p$, there must be a point with only successors satisfying p. Since every point in ω^ω has infinitely many successors, it follows that $V(p)$ must be infinite, hence co-finite. It follows that every world has a successor satisfying p, and therefore, $\mathfrak{F}, V, w \models \Box\Diamond p$. ∎

Of course, this begs the question: which Sahlqvist formulas can safely be combined with pure formulas in the axiomatization of a hybrid logic? A partial answer is provided by Venema [12]. He identified a subclass of the class of modal Sahlqvist formulas, called *simple Sahlqvist formulas*. Using results of Venema, it can be shown that axiomatizations mixing pure and simple Sahlqvist formulas are still complete (cf. [4] for more details).

6 Conclusion

We showed that every extension of the basic hybrid logic with modal Sahlqvist axioms is complete. As a corollary of our approach, we obtained the Beth property for a large class of hybrid logics. Finally, we showed that the new completeness result cannot be combined with the existing completeness result for pure axioms, by providing a modal Sahlqvist formula σ and a pure formula π such that extension of the basic hybrid logic with σ and π is Kripke incomplete.

The situation is radically different in tense hybrid logic. Here the combination problem is not relevant, as every Sahlqvist axiom is expressible as a pure axiom [7].

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