Pure Extensions, Proof Rules, and Hybrid Axiomatics

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Abstract

We examine the role played by proof rules in general axiomatisations for hybrid logic. We prove three main results. First, all known axiomatisations for the basic hybrid language $\mathcal{HL}(@)$ that are strong enough to yield completeness for any pure extension of the minimal logic make use of unorthodox proof rules. We show that this is not accidental: any axiomatisation of the minimal logic that uses only a finite collection of orthodox proof rules will be incomplete for some pure extension. Second, we introduce what we call existential saturation rules and show how they give rise to completeness results for frame classes not definable using pure formulas. Third, we provide an axiomatisation for the stronger language $\mathcal{HL}(@, \downarrow)$ which does not make use of unorthodox rules, but which does yield completeness for any pure extension of the minimal logic, and indeed, for any extension by existential saturation rules.

1 Introduction

The basic hybrid language is the result of enriching ordinary modal logic with *nominals*, a second sort of atomic formula, typically written i, j, k,..., and the @ operator. More precisely, given a countable set of ordinary proposition letters PROP and a countable set of nominals NOM, we define the formulas of the basic hybrid language $\mathcal{HL}(@)$ to be

$$\phi \ ::= \ p \mid i \mid \neg\phi \mid \phi \wedge \psi \mid \Diamond\phi \mid @_i\phi$$

where $p \in$ PROP and $i \in$ NOM. Frames are defined just as in orthodox modal logic, but there is an extra constraint on what counts as a model: in any model, every nominal must be true at a unique world. That is, a model for the basic hybrid language is a pair $(\mathcal{F}, V)$ where $\mathcal{F}$ is a frame and V is a valuation (defined for all symbols in PROP $\cup$ NOM) such that $|V(i)| = 1$ for all $i \in$ NOM.

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Because each nominal is true at a unique world in any model, nominals in effect name worlds. The idea of using 'formulas as terms' in this way is due to Arthur Prior [10, 11]. Moreover, when we have nominals at our disposal, it becomes natural to have a device for evaluating a formula at a named world, and that is the role of the @ operator:

$$(\mathcal{F}, V), w \models @_i\phi \quad \text{iff} \quad (\mathcal{F}, V), u \models \phi, \text{ where } u \text{ is the unique world in } V(i).$$

The nominals and @-operators make it possible to define classes of frames that are not definable in the basic modal language. For example, no formula of the basic modal language can define the class of irreflexive frames, but it is easy to see that

$$@_i\neg\Diamond i$$

is valid on a frame iff the frame is irreflexive (important: valid here means true at all worlds in the frame under all *hybrid* valuations — that is, all valuations which make nominals true at a unique world).

The formula $@_i\neg\Diamond i$ is a simple example of a *pure* formula: it contains no ordinary propositional letters. As is well known, modal formulas containing ordinary proposition letters may well define non-elementary frame classes (the McKinsey formula $\Box\Diamond p \to \Diamond\Box p$ is the simplest example). But any class of frames definable by pure formulas will be an *elementary* class, for if we extend the standard translation (of modal logic to first-order logic) to hybrid logic (see Chapter 7 of [3] for details) nominals correspond to first-order constants. It is therefore reasonable to look for minimal proof systems that are not only complete for the class of all frames, but have the following additional property: whenever *pure* formulas are added as extra axioms, completeness with respect to the (elementary) class of frames defined by these axioms is guaranteed. That is, researchers working on hybrid logic have long been interested in axiomatisations that are automatically complete for *pure extensions*, and a number of such results are known for various hybrid languages (see for example [7, 9, 5]).

This paper takes a closer look at such axiomatisations. We shall prove three main results. First, all known axiomatisations for the basic hybrid language $\mathcal{HL}(@)$ that are complete for pure extensions, make use of unorthodox proof rules (i.e., rules involving syntactic side-conditions). We shall show that this is not accidental: any axiomatisation of the minimal logic (in the language $\mathcal{HL}(@)$) that uses only a finite collection of orthodox proof rules will be incomplete for some pure extension. Second, we introduce what we call existential saturation rules and show how they give rise to completeness results that cover frame classes not definable by pure formulas. Third, we examine what happens when we move from the basic hybrid language $\mathcal{HL}(@)$ to a stronger hybrid language, $\mathcal{HL}(@, \downarrow)$, that has been the focus of much recent work. We shall show that there is an axiomatisation in this language which does not make use of unorthodox rules, but which is complete for pure extensions, and indeed, complete for any extension with existential saturation rules.

2 Unorthodox rules and pure extensions

The following axiomatisation is a simplification of the minimal axiomatisation for the basic hybrid language given in Chapter 7 of [3]:

$\mathcal{HL}(@)$	
Axioms:	
CT	All classical tautologies
$K_\Box$	$\vdash \Box(p \to q) \to \Box p \to \Box q$
$K_@$	$\vdash @_i(p \to q) \to @_i p \to @_i q$
Selfdual$_@$	$\vdash @_i p \leftrightarrow \neg @_i \neg p$
Ref$_@$	$\vdash @_i i$
Agree	$\vdash @_i @_j p \leftrightarrow @_j p$
Intro	$\vdash i \to (p \leftrightarrow @_i p)$
Back	$\vdash \Diamond @_i \phi \to @_i \phi$
Rules:	
MP	If $\vdash \phi$ and $\vdash \phi \to \psi$ then $\vdash \psi$
Subst	If $\vdash \phi$ then $\vdash \phi^\sigma$
Gen$_@$	If $\vdash \phi$ then $\vdash @_i \phi$
Gen$_\Box$	If $\vdash \phi$ then $\vdash \Box \phi$
Name	If $\vdash @_i \phi$ and i does not occur in ϕ, then $\vdash \phi$
BG	If $\vdash @_i \Diamond j \to @_j \phi$ and $j \neq i$ does not occur in ϕ, then $\vdash @_i \Box \phi$

The validity of the axioms should be clear, but two comments on the rules should be made. First, the substitution rule allows us to uniformly replace nominals by nominals and atomic propositions by arbitrary formulas. Second, and more importantly, note that the Name and BG rules are unorthodox: they involve syntactic side conditions of the form X *does not occur in* Y. Let's take a closer look.

First the BG rule. This stands for *Bounded Generalisation*, and can be thought of as a modal analog of the UG (Universal Generalisation) rule of first-order logic. Because j is a nominal distinct from i that does not occur in ϕ, we can read $@_i \Diamond j$ as asserting the existence of a successor (arbitrarily labeled j) of the world labeled i. Accordingly, the antecedent condition of the rule can be read as follows: suppose we can prove that if i has an arbitrary successor j then ϕ holds at j. But then, since j was an *arbitrary* successor of i, the consequent condition of the rule tells us that ϕ must hold at *all* successors of i (that is, $@_i \Box \phi$). The analogy with the first-order rule of universal generalisation should be clear. It's also worth noting that our informal explanation of the BG rule has a natural deduction flavour, and in fact the Box-introduction rule in the natural deduction system for hybrid logic given in [6] is closely related to BG.

What of the Name rule? This tells us that if it is provable that ϕ holds at an arbitrary world i (the world is arbitrary because i does not occur in ϕ) then we can prove ϕ. This rule plays a simple, but crucial, role in the completeness proof given in the following section.

This axiomatisation is sound and complete with respect to the class of all frames. But it is also *complete for all pure extensions*: any extension with pure axioms is complete with respect to the class of frames defined by those axioms (recall that a formula of the basic hybrid language is pure iff it contains no ordinary propositional variables). More precisely, given any set of pure formulas Λ, let $\mathcal{HL}(@)+\Lambda$ denote the above axiomatisation, extended with the axioms in Λ. Then we have:

Theorem 1 (Completeness). *Let Λ be any set of pure axioms. A set of formulas Σ is $\mathcal{HL}(@)+\Lambda$ consistent iff Σ is satisfiable in a model satisfying the frame properties defined by Λ.*

For example, if we choose Λ to be empty set (that is, if we add no additional axioms) then this theorem tells us $\mathcal{HL}(@)$ is complete with respect to the class of all frames. If we choose Λ to be

$$\{@_i\neg\Diamond i, @_i\Diamond j \wedge @_j\Diamond k \rightarrow @_i\Diamond k\}$$

then we have an axiomatisation that is complete with respect to strict partial orders, for the first axiom defines irreflexivity, and the second defines transitivity.

We won't prove Theorem 1 here — a similar result is proved in Chapter 7 of [3], and we shall prove something more general in the following section. Instead we shall address the following question. Theorem 1 covers all pure extensions, but it is based on an axiomatisation that makes use of unorthodox rules (namely Name and BG). Is this necessary, or is the use of such rules avoidable? From [8], we know that if Λ consists of canonical modal formulas, Name and BG are admissible. However, we shall now show that if completeness for arbitrary pure extensions is required, the use of unorthodox rules can only be avoided at the cost of introducing infinitely many rules.

By an orthodox modal rule we mean a rule of the form

$$\frac{\vdash \phi_1(\alpha_1,\ldots,\alpha_n) \quad \& \quad \cdots \quad \& \quad \vdash \phi_k(\alpha_1,\ldots,\alpha_n)}{\vdash \psi(\alpha_1,\ldots,\alpha_n)}$$

Here, $\alpha_1,\ldots,\alpha_n$ range over arbitrary formulas, and are implicitly universally quantified. In the presence of a modus ponens rule (together with enough propositional axioms), we can assume without loss of generality that there is only a single antecedent (a big conjunction), hence all orthodox rules can be assumed to be of the form

$$\frac{\vdash \phi(\alpha_1,\ldots,\alpha_n)}{\vdash \psi(\alpha_1,\ldots,\alpha_n)}$$

In fact, we may assume that ϕ and ψ do not contain any proposition letters, i.e., they are built up from $\alpha_1,\ldots,\alpha_n$ using the Boolean connectives and modal operators. The rank of such a rule will be n. For example, the rank of the $\text{Gen}_\Box$

rule is 1. A rule *preserves validity* on a class of frames F, if for all formulas $\alpha_1, \ldots, \alpha_n$, $\mathsf{F} \models \phi(\alpha_1, \ldots, \alpha_n)$ implies $\mathsf{F} \models \psi(\alpha_1, \ldots, \alpha_n)$. We can now prove the desired result: no finite collection of orthodox rules can be complete for all pure extensions, even if we take as axioms all validities of $\mathcal{HL}(@)$.

Theorem 2. *Let K_h be the set of all formulas in the basic hybrid language that are valid on all frames, let P be a finite set set of orthodox rules, and let L be the axiomatic system formed by taking as axioms K_h, and taking as rules modus ponens, substitution, and all the rules in P. Then there is a pure extension $L + \Lambda$ that is not sound and complete with respect to the class of frames defined by Λ.*

Proof. Let n be the maximal rank of the rules in P — this information is all we need to construct a pure extension that is incomplete with respect to the frame class it defines. Define Λ be the set containing only the the following pure formula:

$$\bigwedge_{1 \leq l \leq 2^n+2} \Diamond i_l \quad \rightarrow \quad \bigvee_{1 \leq k < l \leq 2^n+2} \Diamond(i_k \wedge i_l).$$

$L + \Lambda$ is the axiomatic system L enriched by this single pure axiom (closed under modus ponens, substitution and the rules in P). Let F be the class of frames defined by Λ, that is, the class of all frames in which each world has at most $2^n + 1$ successors. Either the rules in P preserve validity on F or they do not. If they do not, soundness is lost and there is nothing to prove, so assume that the rules P do preserve validity on F. We shall now show that $L + \Lambda$ is *not* complete for F.

Define M to be the class of models based on frames in F. Furthermore, define $\mathcal{F} = (W, R)$ to be the frame where $W = \{1, \ldots, 2^n + 2\}$ and $R = W^2$; clearly, $\mathcal{F} \notin \mathsf{F}$. Finally, let $\mathsf{M}' = \mathsf{M} \cup \{(\mathcal{F}, V) \mid V \text{ is a valuation for } \mathcal{F} \text{ such that } V(i) = V(j) \text{ for all nominals } i, j\}$. Now for the heart of the proof: we shall show that $L + \Lambda$ is sound for the class of models M'.

Claim 1.

- *All formulas in K_h are valid on M'.*
- *Validity on M' is closed under uniform substitution of formulas for propositional variables and nominals for nominals.*
- *Validity on M' is closed under modus ponens*

The proof of Claim 1 is straightforward and is left to the reader.

Claim 2. *All formulas valid on F with at most n propositional variables are valid on M'. Hence $\mathsf{M}' \models \Lambda$.*

Let ϕ be a formula with at most n propositional variables, and suppose for the sake of contradiction that $\mathsf{F} \models \phi$ and $\mathsf{M}' \not\models \phi$. Then there is a valuation V and a world w such that $\mathcal{F}, V, w \Vdash \neg\phi$, and such that V assigns the same world to each nominal. Since only n propositional variables occur in ϕ, and all nominals are true at the same world, it follows that the bisimulation contraction of M (over this restricted vocabulary) has at most $2^n + 1$ worlds; hence, this bisimulation contraction is in F. It follows that $\mathsf{F} \not\models \phi$, which contradicts our initial assumption.

Claim 3. *All rules in P preserve validity on M'.*

Let $\rho \in P$ be a rule

$$\frac{\vdash \phi(\alpha_1, \ldots, \alpha_m)}{\vdash \psi(\alpha_1, \ldots, \alpha_m)}$$

with $m \leq n$, and suppose that $\mathsf{M}' \models \phi(\alpha_1, \ldots, \alpha_m)$ for particular formulas $\alpha_1, \ldots, \alpha_m$. Uniformly substitute $\top$ for each of the propositional variables occurring in $\alpha_1, \ldots, \alpha_m$. We then obtain pure formulas $\beta_1, \ldots, \beta_m$, and by Claim 1 it follows that $\mathsf{M}' \models \phi(\beta_1, \ldots, \beta_m)$. Let $p_1, \ldots, p_m$ be new, distinct propositional variables. Then it follows that

$$\mathsf{M}' \models \phi((p_1 \lhd \phi(p_1, \ldots, p_m) \rhd \beta_1), \ldots, (p_m \lhd \phi(p_1, \ldots, p_m) \rhd \beta_m))$$

where $(\phi \lhd \psi \rhd \chi)$ is shorthand for $(\psi \wedge \phi) \vee (\neg\psi \wedge \chi)$. Hence

$$\mathsf{F} \models \phi((p_1 \lhd \phi(p_1, \ldots, p_m) \rhd \beta_1), \ldots, (p_m \lhd \phi(p_1, \ldots, p_m) \rhd \beta_m))$$

Since F is a purely definable frame class, ρ preserves validity. Hence,

$$\mathsf{F} \models \psi((p_1 \lhd \phi(p_1, \ldots, p_m) \rhd \beta_1), \ldots, (p_m \lhd \phi(p_1, \ldots, p_m) \rhd \beta_m))$$

Since this formula contains at most n propositional variables, it follows by Claim 2 that

$$\mathsf{M}' \models \psi((p_1 \lhd \phi(p_1, \ldots, p_m) \rhd \beta_1), \ldots, (p_m \lhd \phi(p_1, \ldots, p_m) \rhd \beta_m))$$

By closure under uniform substitution (Claim 1), it follows that

$$\mathsf{M}' \models \psi((\alpha_1 \lhd \phi(\alpha_1, \ldots, \alpha_m) \rhd \beta_1), \ldots, (\alpha_m \lhd \phi(\alpha_1, \ldots, \alpha_m) \rhd \beta_m))$$

Recall that $\mathsf{M}' \models \phi(\alpha_1, \ldots, \alpha_m)$. It follows that $\mathsf{M}' \models (\alpha_i \lhd \phi(\alpha_1, \ldots, \alpha_m) \rhd \beta_i) \leftrightarrow \alpha_i$. Hence,

$$\mathsf{M}' \models \psi(\alpha_1, \ldots, \alpha_m)$$

This completes the proof of the third claim, and hence we have shown that $L + \Lambda$ is sound with respect to M'

But now the incompleteness result follows. Consider the following formula

$$\eta = \bigwedge_{1 \leq i \leq 2^n+2} \Diamond p_i \quad \to \bigvee_{1 \leq i < j \leq 2^n+2} \Diamond(p_i \wedge p_j)$$

Notice that $\mathsf{M}' \not\models \eta$. By Claim 1–3, it follows that $L + \Lambda \nvdash \eta$. However $\mathsf{F} \models \eta$, so it follows that $L + \Lambda$ is not complete for F. $\Box$

In short, if we want completeness results that cover arbitrary pure extensions, such as Theorem 1, and we don't want to make use of infinitely many rules, then unorthodox rules (for example, BG and Name) are unavoidable.

3 Existential Saturation Rules

While Theorem 1 is quite general, there are many interesting frame classes that are not definable using pure axioms. One example is the class of frames in which every world has a predecessor, another is the class of Church-Rosser frames. How can we axiomatise such frame classes?

First a simple observation: while neither of these frame classes can be defined in the basic hybrid language, they can be defined in a much stronger hybrid language, namely Prior's original hybrid language which allowed full classical quantification over nominals (see [10, 11]). For example, the class of frames in which every world has a predecessor can be defined in such a language by the formula $\forall s \exists t @_t \Diamond s$, and the class of Church-Rosser frames can be defined by $\forall stu \exists v (@_s \Diamond t \wedge @_s \Diamond u \to @_t \Diamond v \wedge @_u \Diamond v)$.

How does this help? After all, for many purposes we don't want to work in such a strong language (which is essentially a notational variant of the full first-order correspondence language). We want to work in the basic hybrid language. The answer is this: any Priorian formula ξ of the form $\forall s_1 \dots s_n \exists t_1 \dots t_m . \phi$, where ϕ is a formula of $\mathcal{HL}(@)$ that contains no proposition letters and no nominals besides $s_1, \dots, s_n, t_1, \dots, t_m$ gives rise to what we call an *existential saturation rule for* ξ, namely the following (unorthodox) proof rule:

If $\vdash \phi(i_1, \dots, i_n, j_1, \dots, j_m) \to \psi$ then $\vdash \psi$,
provided that $i_1, \dots, i_n, j_1, \dots, j_m$ are distinct and $j_1, \dots, j_m$ do not occur in ϕ.

For example, the existential saturation rule for $\forall stu \exists v (@_s \Diamond t \wedge @_s \Diamond u \to @_t \Diamond v \wedge @_u \Diamond v)$ (the Church-Rosser property) is:

If $\vdash @_i \Diamond j \wedge @_i \Diamond k \to @_j \Diamond l \wedge @_k \Diamond l \to \psi$ *then* $\vdash \psi$,
provided i, j, k, l *are distinct and* l *does not occur in* ψ.

Incidentally, notice the similarity between these existential saturation rules and the proof rules used by [12].

Let ρ be the existential saturation rule for some ξ. We say that a frame *admits* ρ if the set of formulas valid on that frame is closed under ρ. The *class of frames defined by* ρ will be the class of frames defined by ξ.

Lemma 1. *Let ρ be the existential saturation rule for some formula ξ of the form $\forall\vec{s}\exists\vec{t}.\phi$, where ϕ is quantifier-free, pure and nominal free. Every frame satisfying ξ admits ρ.*

Proof. Suppose $\mathfrak{F} \models \forall s_1 \dots s_n \exists t_1 \dots t_n.\phi$, and suppose that the antecedent of ρ is valid on $\mathfrak{F}$, i.e., suppose $\mathfrak{F} \models \phi(i_1, \dots, i_n, j_1, \dots, j_m) \to \psi(i_1, \dots, i_n)$, where $i_1, \dots, i_n, j_1, \dots, j_m$ are distinct. Then $\phi(i_1, \dots, i_n)$ is valid on $\mathfrak{F}$. For, pick and world w and any valuation V. Since $\mathfrak{F} \models \forall\vec{s}\exists\vec{t}.\phi$ and the truth of ψ only depends on the value of the nominals $i_1, \dots, j_n$, we can assume without loss of generality that V assigns worlds to the nominals $j_1, \dots, j_m$ in such a way that ϕ is true. It follows that under this valuation, at the given world, ψ is true. □

This tells us that in order to axiomatise (in the basic hybrid language) frame classes involving properties such as the Church-Rosser property, we can add the relevant existential saturation rules to the axiomatisation while retaining soundness. In fact, we will see that the resulting axiomatisation is *complete* as well.

Given a set of pure axioms Λ and a set of existential saturation rules P, we will use $\mathcal{HL}(@) + \Lambda + P$ to denote the $\mathcal{HL}(@)$ axiomatisation extended with the axioms in Λ and the rules in P.

Theorem 3 (Extended completeness). *Let Λ be a set of pure axioms and let P be a set of existential saturation rules. A set of formulas Σ is $\mathcal{HL}(@)+\Lambda+P$ consistent iff Σ is satisfiable in a model satisfying the frame properties defined by Λ and P.*

Note that Theorem 1 is a special case of this result, namely when P is empty. The remainder of this section is dedicated to the proof of Theorem 3. It closely resembles a Henkin-style completeness proof for first-order logic, with nominals playing the role of first-order constants.

First, we remark that the following validities and rules are derivable:

Lemma 2. *The following are derivable*

$K_@^{-1}$	$\vdash (@_i\phi \to @_i\psi) \to @_i(\phi \to \psi)$
Nom	$\vdash @_i j \to (@_i\phi \leftrightarrow @_j\phi)$
Sym	$\vdash @_i j \to @_j i$
Bridge	$\vdash @_i\Diamond j \wedge @_j\phi \to @_i\Diamond\phi$
Name′	*If* $\vdash i \to \phi$ *then* $\vdash \phi$ *where* i *does not occur in* ϕ
Paste$_\Diamond$	*If* $\vdash @_i\Diamond j \wedge @_j\phi \to \psi$ *and* $j \neq i$ *does not occur in* ϕ *or* ψ, *then* $\vdash @_i\Diamond\phi \to \psi$

Proof. Left to the reader. □

Definition 1. *Let Σ be a set of $\mathcal{HL}(@)$ formulas.*

- Σ *is* named *if one of its elements is a nominal.*

- Σ *is* $\Diamond$-saturated *if for all* $@_i\Diamond\phi \in \Sigma$*, there is a nominal* j *such that* $@_i\Diamond j \in \Sigma$ *and* $@_j\phi \in \Sigma$*.*

- *Let* ρ *be the existential saturation rule corresponding to the strong Priorean formula* $\forall s_1 \cdots s_n \exists t_1 \cdots t_k.\Lambda(s_1, \ldots, s_n, t_1, \ldots, t_m)$*. Then* Σ *is* ρ*-saturated, if for all nominals* $i_1 \ldots i_n$ *there are nominals* $j_1 \ldots j_m$ *such that* $\Lambda(i_1, \ldots, i_m, j_1, \ldots, j_m) \in \Sigma$*.*

Lemma 3 (Lindenbaum Lemma). *Every* $\mathcal{HL}(@) + \Lambda + P$ *consistent set of formulas can be extended to a named,* $\Diamond$*-saturated* $\mathcal{HL}(@) + \Lambda + P$ *MCS, by adding countably many new nominals to the language.*

Proof. Suppose Σ is $\mathcal{HL}(@) + \Lambda + P$ consistent. Let $(i_n)_{n\in\mathbb{N}}$ be an enumeration of a countably infinite set of new nominals, and let $(\phi_n)_{n\in\mathbb{N}}$ be an enumeration of the formulas of the extended language. Let Σ^0 denote $\Sigma \cup \{i_0\}$. The $Name'$ rule guarantees that Σ_0 is consistent. For all $n \in \mathbb{N}$, Σ^{n+1} is defined as follows. If $\Sigma^n \cup \{\phi_n\}$ is $\mathcal{HL}(@) + \Lambda + P$ inconsistent, then $\Sigma^{n+1} = \Sigma^n$. Otherwise:

1. $\Sigma^{n+1} = \Sigma^n \cup \{\phi_n\}$ if ϕ_n is not of the form $@_i\Diamond\psi$.

2. $\Sigma^{n+1} = \Sigma^n \cup \{\phi_n\} \cup \{@_i\Diamond i_m, @_{i_m}\psi\}$ if ϕ_n is of the form $@_i\Diamond\psi$, where i_m is the first new nominal that does not occur in Σ^n or ϕ_n.

Let $\Sigma^\omega = \bigcup_{n\geq 0} \Sigma^n$. Then $\Sigma \subseteq \Sigma^\omega$ and Σ^ω is named, $\Diamond$-saturated, maximal and consistent. The only non-trivial step is in 2., and that this step preserves consistency is guaranteed by the Paste$_\Diamond$ rule. □

Lemma 4 (Rule Saturation Lemma). *Every* $\mathcal{HL}(@) + \Lambda + P$ *consistent set of formulas can be extended to a named,* $\Diamond$*-saturated,* P*-saturated* $\mathcal{HL}(@) + \Lambda + P$ *MCS, by adding countably many new nominals to the language.*

Proof. The proof proceeds in two steps. First, we show that every $\mathcal{HL}(@)+\Lambda+P$ consistent set of formulas Σ can be extended to a set of formulas Σ^+, which is still $\mathcal{HL}(@) + \Lambda + P$ consistent, such that Σ^+ provides witnesses for Σ, in the following sense: for each existential saturation rule $\rho \in P$ corresponding to a strong Priorean formula $\forall s_1 \cdots s_k \exists t_1 \cdots t_m.\Lambda$, and for all nominals $i_1, \ldots, i_k$ occurring in Σ, there are nominals $j_1, \ldots, j_k$ such that $\Lambda(i_1, \ldots, i_k, j_1, \ldots, j_m) \in \Sigma^+$. Such Σ^+ can be constructed as follows.

Let $(i_n)_{n\in\mathbb{N}}$ be an enumeration of a countably infinite set of new nominals, and let $(\rho_n, \vec{i}_n)_{n\in\mathbb{N}}$ be an enumeration of all pairs $(\rho_n, i_{n1} \ldots i_{nk})$ where $\rho_n \in P$ is an existential saturation rule for a strong Priorean formula $\forall s_1 \cdots s_k \exists t_1 \cdots t_m.\Lambda(s_1, \ldots, s_k, t_1, \ldots, t_m)$, and $i_{n1} \ldots i_{nk}$ are nominals occurring in Σ (note that there are at most countably many such pairs). Let $\Sigma^0 = \Sigma$, and for each $n \in \mathbb{N}$, let $\Sigma^{n+1} = \Sigma \cup \{\Lambda(i_{n1}, \ldots, i_{nk}, j_1, \ldots, j_m)\}$, where ρ_n is the existential saturation rule for the strong Priorean formula $\forall s_1 \cdots s_k \exists t_1 \cdots t_m.\Lambda$ and $j_1, \ldots, j_m$ are the first m distinct nominals in the enumeration not occurring in Σ^n. Let $\Sigma^+ = \bigcup_n \Sigma^n$. Then $\Sigma \subseteq \Sigma^+$, Σ^+ is $\mathcal{HL}(@) + \Lambda + P$ consistent and Σ^+ provides witnesses for Σ in the sense described above.

The main argument now runs as follows. Consider any $\mathcal{HL}(@) + \Lambda + P$ consistent set of formulas Γ. Let $\Gamma^0 = \Gamma$ and for all $n \in \mathbb{N}$, let Γ^{n+1} be a $\Diamond$-saturated named MCS extending $(\Gamma^n)^+$ (the Lindenbaum Lemma guarantees there is one). This gives rise to the following chain of consistent sets of formulas:

$$\Gamma = \Gamma^0 \subseteq (\Gamma^0)^+ \subseteq \Gamma^1 \subseteq (\Gamma^1)^+ \subseteq \cdots$$

Let $\Gamma^\omega = \bigcup_n \Gamma^n$. Then Γ^ω is a $\Diamond$-saturated, named P-saturated MCS. Incidentally, during the entire process we expanded the language with only countably many new nominals, and therefore Γ^ω is a countable set. □

Definition 2 (Henkin model obtained from an MCS). *Let Γ be a maximal consistent set of $\mathcal{HL}(@)$ formulas. For all nominals i, let $|i|$ be $\{j \mid @_i j \in \Gamma\}$. Then $\mathfrak{M}_\Gamma = (W, R, V)$ is given by*

$$\begin{array}{rcl} W & = & \{|i| \mid i \textit{ is a nominal}\} \\ |i|R|j| & \textit{iff} & @_i\Diamond j \in \Gamma \\ V(p) & = & \{|i| \in W \mid @_i p \in \Gamma\} \\ V(i) & = & \{|i|\} \end{array}$$

That $\mathfrak{M}_\Gamma$ is well-defined follows from *Ref*, *Sym* and *Nom* (note that *transitivity* is just a special case of *Nom*).

Lemma 5 (Truth Lemma). *For all $\Diamond$-saturated MCSs Γ, nominals i and formulas ϕ, $\mathfrak{M}_\Gamma, |i| \models \phi$ iff $@_i\phi \in \Gamma$*

Proof. By induction on the length of ϕ. □

Lemma 6 (Frame Lemma). *If Γ is a $\Diamond$-saturated, P-saturated $\mathcal{HL}(@)+\Lambda+P$ MCS, then the underlying frame of $\mathfrak{M}_\Gamma$ satisfies the frame properties defined by Λ and P.*

Proof. Since $\mathfrak{M}_\Gamma$ is a named model and Γ contains all instances of elements of Λ, it follows that the underlying frame of $\mathfrak{M}_\Gamma$ validates Λ. Since $\mathfrak{M}_\Gamma$ is a named model and Γ is P-saturated, it follows that the underlying frame of $\mathfrak{M}_\Gamma$ satisfies (the strong Priorean formulas corresponding to) P. □

At this point, we have all the required apparatus in place, and we can finish off the proof by the usual kind of argument.

Proof of Theorem 3. Suppose Σ is $\mathcal{HL}(@) + \Lambda + P$ consistent. By Lemma 4, Σ can be extended to a named, $\Diamond$-saturated, P-saturated MCS Γ. Let $i \in \Gamma$. By Lemma 5, $\mathfrak{M}_\Gamma, |i| \models \Sigma$. By Lemma 6, $\mathfrak{M}_\Gamma$ satisfies all required frame properties. □

A final remark. Existential saturation rules are essentially reflections in the basic hybrid language of certain type of formulas in the stronger Priorean language, namely sentences of the form $\forall s_1 \dots s_n \exists t_1 \dots t_m . \phi(s_1, \dots, s_n, t_1, \dots, t_m)$, where ϕ is quantifier-free, pure and nominal free. A natural question for further work is whether the effects of more complex formulas in the richer language can be captured in a simple way.

4 Axiomatisations for $\mathcal{HL}(@, \downarrow)$

The completeness result just proved can be straightforwardly extended to cover a number of other hybrid languages. Here we sketch how to adapt it to the richer $\mathcal{HL}(@, \downarrow)$. This language, which allows us to bind a nominal to the world of evaluation using the $\downarrow$ binder, has been one of the most extensively explored in contemporary hybrid logic: it corresponds to exactly the generated submodel invariant fragment of first-order logic [1], and once we have $\downarrow$ in our language, it becomes possible to prove very general interpolation results [1, 4]. The language $\mathcal{HL}(@, \downarrow)$ was first axiomatised in [5]. Here we shall improve on this axiomatisation in two ways. First, we will show how to axiomatise $\mathcal{HL}(@, \downarrow)$ by adding a single axiom schema to our axiomatisation to $\mathcal{HL}(@)$. Second, we show how to eliminate the Name and BG rules.

Here is the first axiomatisation. We remark that since we're now dealing with a language with variables and binding, we need to adjust the substitution rule to allow variables and nominals to be substituted for each other, and we need to take the standard precautions to prevent accidental binding of variables. Bearing this in mind, here is the axiomatisation $\mathcal{HL}(@, \downarrow)$-I:

$\mathcal{HL}(@, \downarrow)_{\mathrm{I}}$	
Axioms:	
All axioms for $\mathcal{HL}(@)$	
DA	$\vdash @_i(\downarrow s.\phi \leftrightarrow \phi[s := i])$
Rules:	
All rules for $\mathcal{HL}(@)$	

As promised, our first axiomatisation extends $H(@)$ with a single axiom schema which states the semantics of the $\downarrow$ operator at some arbitrary world named i (the notation $\phi[s := i]$ means substitute the nominal i for all free occurrences of the variable s).

Theorem 4 (Completeness). *Let Λ be a set of pure axioms and let P be a set of existential saturation rules. A set of sentences Σ is $\mathcal{HL}(@, \downarrow)_{\mathrm{I}} + \Lambda + P$ consistent iff Σ is satisfiable in a model satisfying the frame properties defined by Λ and P.*

Proof. Almost no changes to our completeness proof for $\mathcal{HL}(@)$ are required. Once again we build the model out of (equivalence classes of) nominals, and the Lindenbaum Lemma and the Rule Saturation Lemma are unchanged. In fact all we need to do is add an extra clause to our proof of the Truth Lemma for formulas of the form ${\downarrow}s.\psi$, and this is where DA is used. With the Truth Lemma established, completeness follows in the expected way. □

Before going further, a remark on existential saturation rules in $\mathcal{HL}(@,\downarrow)$. First note that for some frame conditions they are no longer needed. For example, the Church-Rosser property, which could not be defined by any pure $\mathcal{HL}(@)$ formula, is defined by the pure $\mathcal{HL}(@,\downarrow)$ formula $\Diamond i \wedge \Diamond j \to @_i(\Diamond{\downarrow}s.@_j\Diamond s)$. However the class of frames in which every world has a predecessor is not definable by means of any pure $\mathcal{HL}(@,\downarrow)$ axiom (since it is not closed under generated subframes) so to axiomatise this frame class we still make use of an existential saturation rule.

Let's turn to our second axiomatisation. While $\mathcal{HL}(@,\downarrow)_{\mathrm{I}}$ is a particularly simple extension of $\mathcal{HL}(@)$, it inherits from $\mathcal{HL}(@)$ the use of the unorthodox Name and BG rules. As we shall now show, if we add a $\mathrm{Gen}_{\downarrow}$ rule, the Name and BG rules can be replaced by axiom schemas:

$\mathcal{HL}(@,\downarrow)_{\mathrm{II}}$	
Axioms:	
All axioms for $\mathcal{HL}(@)$	
DA	$\vdash @_i({\downarrow}s.\phi \leftrightarrow \phi[s := i])$
$\mathrm{Name}_{\downarrow}$	$\vdash {\downarrow}s.(s \to \phi) \to \phi$ provided that s does not occur in ϕ
$\mathrm{BG}_{\downarrow}$	$\vdash @_i\Box{\downarrow}s.@_i\Diamond s$
Rules:	
MP	If $\vdash \phi$ and $\vdash \phi \to \psi$ then $\vdash \psi$
Subst	If $\vdash \phi$ then $\vdash \phi^\sigma$
$\mathrm{Gen}_{@}$	If $\vdash \phi$ then $\vdash @_i\phi$
$\mathrm{Gen}_{\downarrow}$	If $\vdash \phi$ then $\vdash {\downarrow}s.\phi$
$\mathrm{Gen}_{\Box}$	If $\vdash \phi$ then $\vdash \Box\phi$

Theorem 5 (Completeness). *Let Λ be a set of pure axioms and let P be a set of existential saturation rules. A set of sentences Σ is $\mathcal{HL}(@,\downarrow)_{\mathrm{II}} + \Lambda + P$ consistent iff Σ is satisfiable in a model satisfying the frame properties defined by Λ and P.*

Proof. By Theorem 4, it suffices to show that the Name and BG rules are $\mathcal{HL}(@,\downarrow)_{\mathrm{II}}$ derivable. Here's how to derive BG (we leave the derivation of Name to the reader):

$$\dfrac{\dfrac{}{@_i\Box{\downarrow}s.@_i\Diamond s}\,BG_{\downarrow} \qquad \dfrac{\dfrac{\dfrac{@_i\Diamond j \to @_j\phi}{@_j(@_i\Diamond j\to\phi)}\,Gen_@,K_@,Selfdual}{{\downarrow}s.@_i\Diamond s\to\phi}\,DA,Name}{@_i\Box({\downarrow}s.@_i\Diamond s\to\phi)}\,Gen_\Box,Gen_@}{@_i\Box\phi}\,K_\Box,Gen_@,K_@ \qquad \boxed{BG}$$

Thus $\mathcal{HL}(@, \downarrow)_{\mathrm{II}}$ can derive all axioms and rules of $\mathcal{HL}(@, \downarrow)_{\mathrm{I}}$, and hence its completeness follows by Theorem 4. □

5 Conclusion

In this paper we examined the use of proof rules in hybrid axiomatics, and in particular the role they play in establishing completeness results for elementary frame classes. We first showed (Theorem 2) that finite axiomatisations for the basic hybrid language using only orthodox rules cannot be complete for all pure extensions. We then showed (Theorem 3) that existential saturation rules, reminiscent of Venema [12]'s SD rule, allow us to prove completeness results for many elementary frame classes not definable by pure formulas of the basic hybrid language. Finally, we proved a completeness result for the stronger hybrid language $\mathcal{HL}(@, \downarrow)$ which covered all pure extensions, and all extensions by means of existential saturation rules, but which did not involve unorthodox rules. To conclude this paper, let us make some brief remarks about the relation of the work reported here to other recent work on hybrid axiomatics.

Ten Cate, Marx and Viana [8] give an axiomatization of the basic hybrid language that does not make use of unorthodox rules, and that is complete for arbitrary extension with canonical modal axioms. In particular, their result covers the class of Church-Rosser frames, since it is defined by the canonical modal formula $\Diamond\Box p \to \Box\Diamond p$. One might ask whether completeness is also obtained when pure and canonical modal formulas are mixed, for if so, the existential saturation rule for Church-Rosser can always be avoided. The answer is *No* [8]: there is a pure formula ϕ and a canonical modal formula ϕ (in fact, a Sahlqvist formula) such that $\mathcal{HL}(@) + \{\phi, \psi\}$ is not complete. This shows an incompatibility between the Henkin model construction used in this paper and the canonical model method used in [8].

Another recent line of work should also be mentioned. Both the pure extension and the Sahlqvist based results have an obvious limitation: they only apply to elementary, or at least canonical, frame classes. In recent work Bezhanishvili and ten Cate [2] have shown how completeness results for modal logics that admit filtration can be lifted to completeness results for their hybrid versions (the 'hybrid companion logics'). This approach allows completeness results for non-elementary non-canonical logics, such as hybrid versions of **GL** and **Grz**, to be established straightforwardly, without the use of unorthodox rules.

Summing up, at present that are three general techniques for establishing completeness of hybrid logics: via pure extensions, via canonicity, and via transfer from orthodox modal logic. Each handles certain logics the others can't, and mapping the trade-offs between the methods more precisely is an important topic for further research.

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