Nominal Tense Logic

Abstract

This paper considers the logical consequences of making Priorean tense logic referential by sorting its atomic symbols. A second sort of atomic symbol, the *nominal*, is introduced and these are constrained to be true at exactly one point in any model. The resulting gain in expressive power is examined, and a number of logics are axiomatised and shown to be decidable. The relevance of the extension to the semantics of natural language is briefly noted.

This paper investigates a simple method of incorporating temporal reference into Priorean tense logic. A new sort of atomic symbol — the *nominal* — is introduced to languages of tense logic. These new symbols, distinguishable from the ordinary propositional variables, combine with other symbols in the usual way to form wffs. There are no other syntactic changes. The enriched languages are interpreted on frames in the usual way, except that we stipulate that nominals must take the value 'true' at precisely one point in any frame. Nominals are thus 'instantaneous propositions', and the instant at which a nominal is true is the instant it names. We can think of nominals as a mechanism which enables the views of Reichenbach [22] and Prior [20] on tense in natural language to be incorporated in single framework. However while this idea is briefly elaborated on in the concluding remarks, the main aim of the present paper is to discuss the logical properties of Nominal Tense Logic. It turns out that this simple sorting mechanism has a considerable effect on tensed languages: many classes of frames not standardly definable — for example the partial orders, the strict total orders, and the integers — become definable and give rise to richer tense logics.

The paper is structured as follows. After presenting the basic concepts we turn to model theory and examine the increased expressive capabilities. Next we turn to axiomatics and axiomatise both the minimal nominal tense logic and the minimal nominal modal logic. In the following section we prove completeness results for some rather more 'time-like' classes of frames; this is done using Segerberg's [25] cluster manipulation techniques. We also define an algebraic semantics for our languages and prove an algebraic adequacy theorem in the sense of Thomason [28]. Next we consider the question of decidability. We note that because of the new expressive powers of our languages, many logics of interest routinely lack the finite frame property. However a well known theorem of Segerberg's does *not* hold in nominal tense logic: it is possible for a logic to possess the finite model property while lacking the finite frame property. We show how to exploit this using a filtration argument, and thus establish decidability results for a number of logics.

Some historical remarks are in order. The ideas developed here can be traced back to work of Prior and Bull from the late 1960s. In [20, Appendix B] Prior discusses the difficulties of incorporating nominal like entities into tense logic and in [21] he analyses the semantics of 'now' with their help. Somewhat later Bull [5] axiomatised a tense logic with an additional S5 modality in which nominals appear as variables over times which can be bound by quantifiers. It is precisely this last respect that this earlier work differs from the work presented here: Prior and Bull were mostly concerned with more powerful languages in which explicit quantification over nominals could be performed, whereas in this paper they are simply a second sort of atomic symbol. More recently, however, a group of Bulgarian logicians have considered various intensional logics enriched with nominals. They consider nominals in both the setting of propositional dynamic logic [17][11][18], and modal logic [10][12]. As these authors also treat nominals as a second sort of atomic symbol, comparisons can be made with the present paper, and this is done below.

1 Preliminaries

By a language of Nominal Tense Logic (NTL) is meant a selection of two disjoint, countably infinite collections of symbols: NOM $= \{i,\ j,\ k, \ldots\}$, the *nominals* of the language, and VAR $= \{p,\ q,\ r, \ldots\}$, the *propositional variables* of the language. We define ATOM to be NOM $\cup$ VAR and call its elements *atoms*. The wffs of the language are made in the usual way from these atoms using $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\neg$, F, P, H and G. In short, a language of NTL looks just like an ordinary language of tense logic, save for the atomic level: there we have two sorts of atom, nominals and variables. We talk of purely nominal, purely Priorean, and mixed wffs; these are wffs containing only nominals, only variables, or a mixture of the two respectively. For example, $i \rightarrow \neg Fi$ is purely nominal, $FFp \rightarrow Fp$ purely Priorean, and $F(i \wedge p) \wedge F(i \wedge q) \rightarrow F(i \wedge p \wedge q)$ mixed. We often call a language of standard tense logic — that is, a language without nominals — a Priorean language. We use n as a metavariable across nominals and ϕ, ψ, σ and so on, as metavariables across arbitrary wffs. We use the usual syntactic machinery of tense logic: most importantly, $deg(\phi)$ (the *degree* of ϕ) is the number of logical connectives in ϕ and $td(\phi)$ (the *temporal depth* of ϕ) is the maximal level of embedding of tense operators in ϕ. The *mirror image* of a wff ϕ is formed by simultaneously replacing every F by P and G by H, and vice versa.

The semantics of these languages is given in terms of frames and models. As usual, by a *frame* $\mathbf{T}$ is meant a pair $\langle T, < \rangle$ consisting of a nonempty carrier set T and a binary relation $<$ on T. The elements of T are thought of as points of time, and are usually called either points or times, while $<$ is called the precedence relation. We assume the usual tense logical definitions of such concepts as generated subframes and p-morphisms between frames; these concepts are defined in [1] for example. We will be particularly interested in 'time-like' classes of frames, thus we shall pay special attention to: the *strict partial orders* (SPOs), which consists of those frames in which the precedence relation is both transitive and irreflexive; the *strict total orders* (STOs), which are those SPOs which are also trichotomous; the *partial orders* (POs), which consists of those frames in which the precedence relation is transitive, reflexive and antisymmetric; and the *total orders* (TOs), which are trichotomous POs, that is, chains.

By a *model* $\mathbf{M}$ is meant a pair $\langle \mathbf{T}, V \rangle$ where $\mathbf{T}$ is a frame, and V a *valuation* on $\mathbf{T}$. We say that $\mathbf{T}$ is the frame underlying $\mathbf{M}$, and often call the points of the underlying frame the points of the model. Most of our terminology concerning models is standard: indeed it is only in the definition of what it is to be a valuation that NTL differs from standard tense logic. As usual, a valuation on $\mathbf{T}$ is a mapping from the atoms of our language to $Pow(T)$, but we place a restriction on the subsets of T that nominals may be assigned. *Nominals must always be assigned singleton subsets of a frame.* A mapping from the atoms to $Pow(t)$ that does not obey this constraint is *not* a valuation. As usual, propositional variables can denote arbitrary subsets of T. With this one change made, everything proceeds as in standard tense logic. In particular, we define the truth of a wff ϕ at a point t of a model $\mathbf{M}$, $\mathbf{M} \models \phi[t]$ in the usual fashion. We say that a wff ϕ is valid in a model $\mathbf{M}$ (written $\mathbf{M} \models \phi$) iff for all points $t \in \mathbf{M}$, $\mathbf{M} \models \phi[t]$; and we say that ϕ is valid on a frame $\mathbf{T}$ (written $\mathbf{T} \models \phi$) iff for all valuations V on $\mathbf{T}$, $\langle \mathbf{T}, V \rangle \models \phi$.

We will frequently talk of *paths* in what follows. By a path through a frame $\langle T, < \rangle$ is meant any finite sequence of elements of T such that for every pair t_m, t_{m+1} of the sequence, either $t_m < t_{m+1}$, or $t_{m+1} < t_m$. That is, a path through a frame is a sequence of moves both forward and backward in time. Sometimes to emphasize the bidirectionality of the concept we refer to paths as zig-zag paths. By the length of a path is meant the sequence length. A frame is connected iff there is a path between any two of its points.

Many analogs of results for purely Priorean languages hold for languages of NTL; for example, isomorphic frames are equivalent. Another standard result which is sometimes useful is the following. Let $\mathbf{T} = \langle T, < \rangle$ be a frame and $t \in T$. By $S_n(\mathbf{T}, t)$, the *n-hull around* t, is meant the set of all points of $\mathbf{T}$ that are related in n steps to t. The Horizon Lemma states that for any frame $\mathbf{T}$ and any two valuations V, V' on $\mathbf{T}$ such that $V(a) \cap S_n(\mathbf{T}, t) = V'(a) \cap S_n(\mathbf{T}, t)$ for all atoms a, $\langle \mathbf{T}, V \rangle \models \phi[t]$ iff $\langle \mathbf{T}, V' \rangle \models \phi[t]$, for all ϕ such that $td(\phi) \leq n$.

More importantly, filtration theory [25] adapts straightforwardly to languages of NTL:

Definition 1.1 (Filtrations) Let $\mathbf{M} = \langle \mathbf{T}, V \rangle$ be a model and Σ a set of wffs closed under subformulas. Define an equivalence relation $\sim$ on T by $t \sim t'$ iff for all $\sigma \in \Sigma$ and $t, t' \in T$, $\mathbf{M} \models \sigma[t]$ iff $\mathbf{M} \models \sigma[t']$. Let $E(t)$ denote the equivalence class of t. Define $F = \{E(t) : t \in T\}$. Now suppose that $<_f$ is a binary relation on F satisfying:

1. $s < t \;\Rightarrow\; E(s) <_f E(t)$
2. $E(s) <_f E(t) \;\Rightarrow\; ((F\sigma \in \Sigma \;\&\; \mathbf{M} \models \sigma[t]) \;\Rightarrow\; \mathbf{M} \models F\sigma[s])$
3. $E(s) <_f E(t) \;\Rightarrow\; ((P\sigma \in \Sigma \;\&\; \mathbf{M} \models \sigma[s]) \;\Rightarrow\; \mathbf{M} \models P\sigma[t])$.

and further suppose that $V_f : \mathrm{ATOM} \longrightarrow Pow(F)$ is a function satisfying:

1. $E(t) \in V_f(p)$ iff $t \in V(p)$, for all $p \in VAR_{\mathcal{L}}$
2. $E(t) \in V_f(i)$ iff $t \in V(i)$, for all $i \in \Sigma \cap NOM$
3. $V_f(i)$ is a singleton subset of F, for all $i \in NOM \backslash \Sigma$.

Then $\mathbf{M}^f = \langle \langle F, <_f \rangle, V_f \rangle$ is called a *filtration of* $\mathbf{M}$ *through* Σ. □

It follows by the standard arguments (see [25]) that filtrations exist. Moreover, as usual, given a transitive frame it is always possible to form a filtration that is transitive by means of the following definition:

$$\begin{array}{lll} E(s) <_f E(t) & \text{iff} & ((F\sigma \in \Sigma \;\&\; \mathbf{M} \models \sigma \vee F\sigma[t]) \;\Rightarrow\; \mathbf{M} \models F\sigma[s], \text{ and} \\ & & ((P\sigma \in \Sigma \;\&\; \mathbf{M} \models \sigma \vee P\sigma[s]) \;\Rightarrow\; \mathbf{M} \models P\sigma[t]. \end{array}$$

Following Segerberg [25] we call such filtrations Prior filtrations.

However although many filtrations thus exist, we haven't yet checked that filtrations are models; that is, we haven't yet checked that V_f in the above definition assigns singletons to nominals. This we will now do.

Lemma 1.1 (Filtrations are models) Let $\mathbf{M}$ be a model, Σ a set of wffs closed under subformulas, and $\mathbf{M}^f$ a filtration of $\mathbf{M}$ through Σ. Then $\mathbf{M}^f$ is a model.

Proof:

We need merely check that V_f is a valuation. For the propositional variables the definition is unproblematic, as propositional variables may denote arbitrary subsets. So next consider the nominals. Suppose $i \in \Sigma$, and that $V(i) = \{t\}$. Clearly by the 'if' direction of the second clause for V_f there is at least one point of F in $V_f(i)$, namely $E(t)$. Equally clearly, by 'only if' direction of the same clause, there is no other; otherwise we would have that $V(i)$ contained more than one element, and as V is a valuation this is impossible. Thus V_f handles all the nominals in Σ correctly, and by definition V_f 'freely assigns' singletons to any nominals not in Σ, hence V_f is a valuation and $\mathbf{M}^f$ a model. $\Box$

With this observation the way is cleared for an NTL version of a standard result. By induction on $deg(\phi)$ we establish:

Theorem 1.1 (Filtration theorem) Let $\mathbf{M} = \langle\langle T, <\rangle, V\rangle$ be a model, Σ a set of wffs closed under subformulas, and $\mathbf{M}^f = \langle\langle F, <_f\rangle, V_f\rangle$ a filtration of $\mathbf{M}$ through Σ. Then:

$$\mathbf{M} \models \sigma[t] \ \text{ iff } \ \mathbf{M}^f \models \sigma[E(t)],$$

for all $\sigma \in \Sigma$, and all $t \in T$. $\Box$

Filtrations play an important role in the work that follows. Indeed they already tell us that the validities of NTL form a recursive set. The argument is standard: given a wff σ that is *not* valid let Σ be the set of all its subformulas. Suppose $card(\Sigma) = n$. Then filtrating any model that falsifies σ through Σ yields a model of cardinality at most 2^n that also falsifies σ. In short we have an upper bound on the size of model needed to falsify wffs, thus searching through all finite models up to the size limit is an algorithm for deciding validity.

However there are also many differences between NTL and Priorean Tense Logic (indeed this paper is essentially an exploration of these differences). Firstly note that *uniform substitution* is not a sound rule of inference: from the fact that $\mathbf{T} \models \phi$ we *cannot* infer that $\mathbf{T} \models \phi^s$, where ϕ^s results from ϕ by uniformly substituting formulas for atoms. To give the simplest possible counterexample, $\mathbf{T} \models i$ iff $\mathbf{T}$ is a frame of cardinality one, but no such frame (and in fact no frame at all) validates p, the result of uniformly substituting p for i. The loss of substitution as a sound rule of inference is a direct outcome of the constraint we have put on what functions count as valuations. To put it another way, the basic idea we are investigating in this paper is what happens when the atomic information slots are 'syntactically marked' as bearers of a certain type of information. In NTL we have two such slots: propositional variable slots (which bear arbitrary information) and nominal slots (which bear single point of time information). Unrestricted substitution destroys this marking. Instead we should consider only those substitutions which respect our sortal restrictions. Thus we define: *a wff ψ is obtained from a wff ϕ by* NTL substitution *iff ψ is the result of uniformly substituting arbitrary* NTL *wffs θ for propositional variables in ϕ, and uniformly substituting nominals for nominals in ϕ.* Clearly this is a sound rule of inference.

But the differences run deeper. For example, although we are free to use the filtration technique, the method of *unravelling* [24, pages 124–127], fails to transfer to languages of NTL. In Priorean tense logic, unravelling turns a (Priorean) model based on a frame of arbitrary structure into an equivalent (Priorean) model based on a tree. Among other things this shows that the purely Priorean validities on the class of all frames are precisely the same as the purely Priorean validities on the class of intransitive frames: purely Priorean languages cannot 'see' intransitivity. However NTL can. Consider the following wff:

$$FFi \to \neg Fi.$$

As a simple check shows, this is valid on precisely the intransitive frames. Thus unravelling destroys structure that nominals can see. This is the first example we have seen of a difference in expressive power that exists between the two sorts; we will see many more in the work that follows.

We later briefly discuss languages of nominal modal logic (NML). The definition of their syntax and semantics is the expected one: again we have two sorts of atom, namely nominals and variables, and again nominals denote singleton subsets of frames. By a *modal path* through a frame $\langle T, < \rangle$ is meant a finite sequence of elements of T such that for all pairs t_m and t_{m+1} of the sequence, $t_m < t_{m+1}$. Note that modal paths are unidirectional.

2 Model Theory

We say an NTL formula ϕ *defines* a class of frames $\mathcal{T}$ iff: $\mathbf{T} \models \phi$ iff $\mathbf{T} \in \mathcal{T}$. For example, we have just noted that $FFi \rightarrow \neg Fi$ defines the intransitive frames. Note that if ϕ defines $\mathcal{T}$ and ψ defines $\mathcal{T}'$, then $\phi \wedge \psi$ defines $\mathcal{T} \cap \mathcal{T}'$. For further discussion of definability in standard tense and modal logic see [1] or [2].

None of the following classes of frames are definable in a purely Priorean language: the irreflexive, asymmetric, antisymmetric, trichotomous, (right) directed or (right) discrete frames. Furthermore, neither the SPOs, STOs, POs or TOs are definable in a purely Priorean language. However it is straightforward to verify that each of the first six classes of frames is defined by the purely nominal wff given:

$$
\begin{array}{ll}
i \rightarrow \neg Fi & \forall x \neg(x < x) \\
i \rightarrow \neg FFi & \forall xy\, x(x < y \;\rightarrow\; \neg x < y) \\
i \rightarrow G(Fi \rightarrow i) & \forall xy((x < y \;\wedge\; x < y) \;\rightarrow\; x = y) \\
Pi \vee i \vee Fi & \forall xy(x < y \;\vee\; x = y \;\vee\; x < y) \\
FPi & \forall xy \exists z(x < z \;\wedge\; y < z) \\
i \rightarrow (F\top \rightarrow FHH\neg i) & \forall xy(x < y \;\rightarrow\; \exists z(x < z \;\wedge\; \neg \exists w(x < y < z)))
\end{array}
$$

(Corresponding to right directedness and discreteness are left directedness and discreteness, defined in the obvious way by mirror images. We regard $\top$ as shorthand for $i \vee \neg i$.)

Next note that $FFi \rightarrow Fi$ defines transitivity and $i \rightarrow Fi$ defines reflexivity. Thus by making of these two wffs, together with wffs drawn from the previous list, we can define the SPOs, STOs, POs and TOs by means of purely nominal wffs. For example, let ϕ^{STO} be

$$(i \rightarrow \neg Fi) \wedge (Pi \vee i \vee Fi) \wedge (FFi \rightarrow Fi).$$

This purely nominal wff defines the STOs.

With mixed wffs we can do more: we can define both the integers and the natural numbers up to isomorphism. Define ϕ^Z to be:

$$\phi^{\mathrm{STO}} \wedge (H(Hp \rightarrow p) \rightarrow (PHp \rightarrow Hp)) \wedge (G(Gp \rightarrow p) \rightarrow (FGp \rightarrow Gp))$$

We then have that $\mathbf{T} \models \phi^Z$ iff $\mathbf{T} \cong \mathbf{Z}$. To see this note that in [1, page 163] van Benthem shows that the two purely Priorean conjuncts of ϕ^Z define $\mathbf{Z}$ on the class of *connected strict partial orders*. But ϕ^{STO} restricts us to this class.

Next define ϕ^N to be:

$$\phi^{\mathrm{STO}} \wedge (H(Hp \rightarrow p) \rightarrow Hp) \wedge (G(Gp \rightarrow p) \rightarrow (FGp \rightarrow Gp))$$

Again by appeal to a result of van Benthem's we have $\mathbf{T} \models \phi^N$ iff $\mathbf{T} \cong \mathbf{N}$.

It is unavoidable that both ϕ^Z and ϕ^N are *mixed* sentences. We will shortly see that only first order classes of frames are definable using purely nominal sentences, thus it follows that no purely nominal sentence can uniquely define these structures. Further, van Benthem's results concerning the definability of these structures in Priorean languages are 'best possible' results for purely Priorean languages, as the preservation of purely Priorean validity under the formation of p-morphic images and disjoint unions prevents the definition of either $\mathbf{Z}$ or $\mathbf{N}$ using just propositional variables. The mixture of nominals and propositional variables is thus necessary.

All initial segments of $\mathbf{N}$ are also definable. (They are not in a purely Priorean language.) Define ϕ^{L^n} to be

$$\phi^{\text{STO}} \wedge G^n \bot \wedge (F^{n-1}\top \vee PF^{n-1}\top),$$

where $n \in N$ such that $n \geq 1$. Then $\mathbf{T} \models \phi^{L^n}$ iff $\mathbf{T}$ is a STO of length exactly n.

Next, in NTL we can demand that every point has exactly n successors. First, using either propositional variables or nominals we can insist that every point has *at most* n successors, as the following encoding of the Pigeonhole Principle shows:

$$\bigwedge_{1 \leq \alpha \leq n+1} Fa_\alpha \rightarrow \bigvee_{1 \leq \alpha \leq n;\ 2 \leq \beta \leq n+1;\ \alpha \leq \beta} F(a_\alpha \wedge a_\beta).$$

(Here the a_α are distinct atoms: either nominals or variables or a mixture can be used.) However using only variables we cannot insist that every point has *at least* n successors. With nominals, on the other hand, we need merely write down:

$$F\top \wedge (\bigwedge_{1 \leq \alpha < n} Fi_\alpha \rightarrow F \bigwedge_{1 \leq \alpha < n} \neg i_\alpha)$$

where the i_α are distinct nominals.

What can we say of a more general nature? For purely Priorean languages there are four classic validity preservation results: validity is preserved under the formation of generated subframes, disjoint unions, and p-morphic images; and anti-preserved under the formation of ultrafilter extensions (for the definitions of these concepts we once again refer the reader to [1]). As languages with nominals are more expressive than Priorean languages, we might expect that one or more of these preservation results will fail. This is precisely what happens: for NTL only the generated subframe and ultrafilter extension results still hold.

Let's first briefly examine the two preservation results we retain. The anti-preservation of validity under ultrafilter extensions remains because $ue(V)(i)$ will contain only the principle ultrafilter generated by $V(i)$, for every nominal i and every valuation V; thus $ue(V)$ assigns singletons to nominals and is a valuation. With this noted, the usual proof of the anti-preservation result proceeds unchanged. In the generated subframe case we need to be a little careful in formulating what we mean by a generated submodel of $\langle \mathbf{T}, V \rangle$ — not every pair $\langle \mathbf{S}, V{\downarrow_S} \rangle$, where $\mathbf{S}$ is a generated subframe of $\mathbf{T}$ and $V{\downarrow_S}$ the restriction of V to $\mathbf{S}$, is a model as $V{\downarrow_S}$ may assign $\emptyset$ to nominals — but we need merely confine our attention to pairs where this does not happen. The usual induction then gives a generated submodel theorem for languages with nominals; and as an immediate corollary we have that validity is transmitted from any frame to its generated subframes.

The two results that fail are more interesting. For Priorean languages we have that given an indexed collection of frames $\{\mathbf{T}_m : m \in M\}$, if for all $m \in M$ $\mathbf{T}_m \models \phi$, then $\biguplus \mathbf{T}_m \models \phi$. An immediate consequence of this result is that Priorean languages cannot define the universal relation $\forall xy(x < y)$. Another obvious consequence is that connectedness is not definable in a Priorean language; indeed something stronger holds — no purely Priorean definable class of frames consists solely of connected frames.

For languages containing nominals the preservation result no longer holds. An immediate counterexample is given by the class of trichotomous frames, defined by $Pi \vee i \vee Fi$. Another is provided by the class of (right or left) directed frames. Yet another is given by the universal relation; this condition *is* definable using nominals, by Fi.

Now, although the disjoint union preservation result fails for languages with nominals, a little reflection shows that it 'only just' fails. Suppose we have two frames $\mathbf{T}_1$ and $\mathbf{T}_2$ on each of which ϕ is valid. To keep things simple suppose ϕ contains occurrences of only one nominal, say i. We know that we cannot conclude that $\mathbf{T}_1 \uplus \mathbf{T}_2 \models \phi$, but why not? The reason is that in any valuation on $\mathbf{T}_1 \uplus \mathbf{T}_2$, on one of the components, say $\mathbf{T}_1$, i will be false everywhere. This is a situation that the validity of ϕ on the component frames simply gives us no information about: in any valuation on either frame i is true *somewhere*.

But suppose we knew something more: namely that not only was ϕ valid on each frame, but $\phi[\perp/i]$ was also. Then, intuitively, we *would* have the information needed to guarantee validity on the disjoint union: the validity of the new formula blocks the possibility that i being false everywhere in a component will cause trouble. This is indeed the case: indeed, not only is the condition *sufficient*, it is also *necessary* as long as the disjoint union is not trivial — that is, as long as at least two frames are stuck together.

To state the result in full generality we need merely extend the above intuitions to the case where ϕ contains many different nominals. Essentially all we need to do is account for all the different ways the nominals can be 'dealt out' — like cards from a pack — to the 'players' — the components of the disjoint union. (A particular deal, of course, is just a valuation.) That is, we must take into account all possible uniform substitutions of $\perp$ for nominals in ϕ. Let $S^{\perp}(\phi)$ be the (finite) set consisting of precisely all the possible sentences obtainable by uniformly substituting $\perp$ for nominals occurring in ϕ, including the null substitution. For example, $S^{\perp}(i \wedge Fj) = \{i \wedge Fj,\ i \wedge F\perp,\ \perp \wedge Fj,\ \perp \wedge F\perp\}$. Let $\phi^{\perp}$ denote the conjunction of these sentences. Then we have:

Theorem 2.1 Let $\{\mathbf{T}_m : m \in M\}$ be a family of frames such that $card(M) \geq 2$. Then:

$$\biguplus \mathbf{T}_m \models \phi \text{ iff } \forall m \in M\ \mathbf{T}_m \models \phi^{\perp}$$

for all wffs ϕ.

Proof:

A straightforward argument using the generated submodel result. Use the fact that nominals assigned points outside a generated subframe $\mathbf{S}$ behave like $\perp$ on $\mathbf{S}$. □

While p-morphisms preserve validity for Priorean languages, they do not do so for languages with nominals. There are many obvious counterexamples. Note that the unique function from $\mathbf{Z}$ to the singleton reflexive frame $\langle\{0\}, \{\langle 0,0\rangle\}\rangle$ is a p-morphism; but both $i \rightarrow \neg Fi$ and $i \rightarrow \neg FFi$ are valid on $\mathbf{Z}$ and invalid on the singleton reflexive loop. A p-morphism is constructed in [1, pages 160–161] where the source frame is discrete, and the target frame indiscrete, thus demonstrating that discreteness is not Priorean definable. But we know that discreteness is definable with nominals, hence van Benthem's construction provides yet another counterexample.

For *models* however, the p-morphic link is the correct one. That is, if f is a p-morphism from $\mathbf{M}_s = \langle \mathbf{S}, V_s \rangle$ to $\mathbf{M}_t = \langle \mathbf{T}, V_t \rangle$, then we still have that

$$\mathbf{M}_s \models \phi[s] \text{ iff } \mathbf{M}_t \models \phi[f(s)],$$

for all $s \in S$ and all wffs ϕ, as the usual induction on $deg(\phi)$ shows. Note why we cannot derive from this the usual validity preservation result. Suppose f is a p-morphism from $\mathbf{S}$ to $\mathbf{T}$, and suppose $\langle \mathbf{T}, V \rangle \not\models \phi[t]$. However $f^{-1}[V(i)]$ may not be a singleton subset of $\mathbf{S}$, thus we cannot always transfer the falsifying valuation to $\mathbf{S}$.

Let us turn to the correspondence between languages of NTL and classical languages. Following [2] we define $\mathbf{L}_0$ to be a first order language with identity that contains precisely one non-logical symbol, a binary relation symbol '$<$'. Note that any frame is a structure for this language. Now, only $\mathbf{L}_0$ expressible classes of frame are definable by purely nominal sentences. To see this note that to deal with nominals we need merely augment the *standard translation* [1, page 151] of tensed languages into classical languages by adding the clause that the standard translation of any nominal i, $ST(i)$, is to be the $\mathbf{L}_0$-wff $x_i = t$. (Here x_i is the $\mathbf{L}_0$ variable designated as corresponding to the nominal i, and t the $\mathbf{L}_0$ variable representing the point of evaluation.) Now saying that a purely nominal formula ϕ is valid on a frame $\mathbf{T}$ is equivalent to saying that $\forall t \forall x_{i_1} \cdots x_{i_n} ST(\phi)$ is true in any first order model based on the structure $\mathbf{T}$, where the $x_{i_1}, \ldots, x_{i_n}$ correspond to all the nominals in ϕ. But $\forall t \forall x_{i_1} \cdots x_{i_n} ST(\phi)$ is a first order sentence, in fact an $\mathbf{L}_0$ sentence. With purely Priorean languages we need second order quantification when we talk about validity — propositional variables correspond to predicates, thus quantification over predicates is required to capture the effect of varying valuations. With nominals matters are simpler. This translation immediately yields a number of results: that nominal validity is r.e., compactness and Löwenheim-Skolem theorems, and so on. Next consider *frame consequence*. This is defined as follows: ϕ is a frame consequence of a set of wffs Σ (written $\Sigma \models_f \phi$) iff whenever Σ is valid on a frame $\mathbf{T}$, so is ϕ. As is well known [27] this relation is not r.e. for purely Priorean languages. However it is r.e. for purely nominal sets of sentences, as for such sentences $\Sigma \models_f \phi$ iff $ST(\Sigma) \models ST(\phi)$. That is, for nominals frame consequence $\models_f$ reduces to the r.e. relation of first order consequence, $\models$.

We now know that only $\mathbf{L}_0$ conditions are definable using purely nominal wffs. We also know that some of these conditions — such as irreflexivity — are not definable in any purely Priorean language. Now, Priorean languages, because they can define higher order conditions, can define conditions no purely nominal language can — but can they define any first order conditions that purely nominal languages cannot? So far we have seen no counterexamples; could it be that as far as $\mathbf{L}_0$ conditions are concerned, purely nominal languages are stronger than purely Priorean ones? The answer is no, but some work is required to see this. The counterexample that follows and the idea underlying the proof are due to Johan van Benthem.

The counterexample is 'transitivity plus atomicity'. An atomic frame is one in which every point x precedes an 'atom' y that is its own only successor:

$$\forall x \exists y (x < y \wedge \forall z (y < z \rightarrow z = y));$$

and the class of all frames that are both transitive and atomic are definable in purely Priorean languages by the conjunction of $FFp \rightarrow Fp$ (4) with $GFp \rightarrow FGp$ (McKinsey's axiom). However no purely nominal wff can define this condition. The essence of the argument that follows is this: any such formula ϕ which putatively defines this condition can be falsified on $\mathbf{N}$ (the natural numbers in their usual order) as $\mathbf{N}$ contains no atoms. Because of a certain 'stability property' which we will demonstrate, it is possible to by means of filtration to turn this falsifying model into a transitive and atomic falsifying model, showing that no such wff can define the desired class.

First the 'stability lemma'. Its intuitive content is this: given any purely nominal wff ϕ and a valuation on $\mathbf{N}$, by moving sufficiently far to the right along $\mathbf{N}$ we reach a point where the truth values of ϕ and all its subformulas stabilise to some fixed values. This is because eventually we reach a point where all nominals in ϕ denote points in the past. The only tricky part in establishing this is driving through the clause for formulas of the form $P\phi$, as such formulas can look back at points before things settled down; this motivates the use of the *temporal depth* measure in the following lemma:

Lemma 2.1 Let Σ be a set of sentences closed under subformulas such that the only atoms in Σ are a finite collection of nominals. Let V be a valuation on $\mathbf{N}$, and let $l-1$ be the largest natural number that V assigns to some nominal in Σ. That is, $l-1 = max \bigcup_{i \in \Sigma} V(i)$. Then for all wffs $\sigma \in \Sigma$, for all $n > l + td(\sigma)$, $\langle \mathbf{N}, V \rangle \models \sigma[n]$ iff $\langle \mathbf{N}, V \rangle \models \sigma[l + td(\sigma)]$.

Proof:

Induction on $td(\sigma)$. Suppose $td(\sigma) = 0$. Then as σ is a purely nominal sentence it is either a nominal or a boolean combination of nominals, all of which are in Σ. As all nominals in Σ are false from l onwards, the result is clear by induction on $deg(\sigma)$.

Assume the result holds for all $\sigma' \in \Sigma$ such that $td(\sigma') < m$, where $m > 0$. Suppose $td(\sigma) = m$. We want to show that for all $n > l + m$, $\langle \mathbf{N}, V \rangle \models \sigma[n]$ iff $\langle \mathbf{N}, V \rangle \models \sigma[l + m]$.

Suppose σ has the form $P\psi$. Clearly $\langle \mathbf{N}, V \rangle \models \sigma[l + m]$ implies $\langle \mathbf{N}, V \rangle \models \sigma[n]$. So suppose that $\langle \mathbf{N}, V \rangle \not\models \sigma[l + m]$. Then for all $h < l + m$ $\langle \mathbf{N}, V \rangle \models \sigma[h]$, which means in particular that $\langle \mathbf{N}, V \rangle \models \sigma[l + (m-1)]$. As $td(\psi) = m - 1$, by the inductive hypothesis we have that for all $n > l + m$, $\langle \mathbf{N}, V \rangle \not\models \sigma[n]$ which means that ψ is false everywhere on $\mathbf{N}$. Thus trivially for all $n > l+m$, $\langle \mathbf{N}, V \rangle \not\models \sigma[n]$ as required. Alternatively, if we assume that σ has the form $F\psi$ a similarly styled argument also gives the required result.

The only other possibility is that σ is a boolean combination of elements $\sigma'_1, \ldots, \sigma'_k$ of Σ such that $td(\sigma'_j) < m$ or $td(\sigma'_j) = m$ and σ'_j has the form $P\psi$ or $F\psi$. But our argument so far tells us that for all such σ'_j $(1 \leq j \leq k)$, $\langle \mathbf{N}, V \rangle \models \sigma_j[n]$ iff $\langle \mathbf{N}, V \rangle \models \sigma'_j[l + td(\sigma'_j)]$, for all $n > l + td(\sigma'_j)$, and as σ is a boolean combination of such forms an easy inductive argument shows that $\langle \mathbf{N}, V \rangle \models \sigma[l + m]$ iff $\langle \mathbf{N}, V \rangle \models \sigma[n]$, for all $n > l + m$. $\Box$

Theorem 2.2 If a purely nominal wff ϕ is falsifiable on $\mathbf{N}$, then ϕ is falsifiable in a (finite) transitive and atomic model.

Proof:

Let ϕ be a purely nominal wff such that for some valuation V' and point $k \in N$, $\langle \mathbf{N}, V' \rangle \not\models \phi[k]$. Let Σ^- be the smallest set of sentences containing ϕ that is closed under subformulas, and let $l - 1$ be the largest natural number that V' assigns to any nominal in Σ^-. Let $td(\phi) = c$. As for all $\sigma \in \Sigma^-$ $td(\sigma) \leq c$ we know by the previous lemma that for all $n > l + c$ the truth values of sentences in Σ are stable in $\langle \mathbf{N}, V' \rangle$.

Let j be any nominal not occurring in Σ^-, and V be the valuation that is just like V' save possibly that $V(j) = \{(l + c) - 1\}$. As V and V' agree on the values of all atoms in Σ^-, we have that for all $\sigma \in \Sigma^-$, for all $n > l + c$, $\langle \mathbf{N}, V \rangle \models \sigma[n]$ iff $\langle \mathbf{N}, V \rangle \models \sigma[l + c]$, and moreover $\langle \mathbf{N}, V \rangle$ also falsifies ϕ at k. Let $\Sigma = \Sigma^- \cup \{Fj, j\}$. Note that Σ is a finite set of sentences closed under subformulas. Clearly for all $\sigma \in \Sigma$, for all $n > l + c$, $\langle \mathbf{N}, V \rangle \models \sigma[n]$ iff $\langle \mathbf{N}, V \rangle \models \sigma[l + c]$.

Take a Prior filtration of $\langle \mathbf{N}, V \rangle$ through Σ to form $\mathbf{M}^f$. $\mathbf{M}^f$ falsifies ϕ at $E(k)$ by the Filtration Theorem; $\mathbf{M}^f$ is transitive because we took a Priorean filtration; and $\mathbf{M}^f$ is finite because Σ was finite. If we can show that $\mathbf{M}^f$ is atomic we are through.

We know that in the model $\langle \mathbf{N}, V \rangle$, the truth values of all wffs $\sigma \in \Sigma$ are stable from $l + c$ onwards, but this means that all $n \geq l + c$ are in the same equivalence class. Call the element of $\mathbf{M}^f$ of which they are all a member $E(l + c)$. We now show that $E(l + c)$ is an atom that all other elements of $\mathbf{M}^f$ precede.

In any filtration whatsoever, if $t < t'$ in the original model then $E(t) <_f E(t')$ in the filtration. Hence any other element $E(t) \in \mathbf{M}^f$ must precede $E(l + c)$. Moreover it follows from the definition of $<_f$ coupled with the stability of the truth values of the wffs in Σ that $E(l+c) <_f E(l+c)$. Thus the only thing that could prevent $E(l+c)$ from being the desired atom would be if $E(l + c)$ preceded some distinct $E(t) \in \mathbf{M}^f$. We now show that this is impossible.

Suppose $E(t) \neq E(l+c)$ and $E(l+c) <_f E(t)$. We need merely note that no $h \in N$ such that $h \leq (l+c)-1$ can be in $E(t)$. For if $h < (l+c)-1$ then $\langle \mathbf{N}, V \rangle \models Fj[h]$, and if $h = l-1$ then $\langle \mathbf{N}, V \rangle \models j[h]$, and as both Fj and j are in Σ we have by the Filtration Theorem that $\mathbf{M}^f \models Fj[E(t)]$ or $\mathbf{M}^f \models j[E(t)]$, which by the definition of $<_f$ would mean that $\mathbf{M}^f \models Fj[E(l+c)]$. But another appeal to the Filtration Theorem shows that this in turn would mean that $\langle \mathbf{N}, V \rangle \models Fj[l+c]$, which is impossible as $V(j) = \{(l+c)-1\}$. In short, any such $E(t)$ would be empty, so $E(l+c)$ precedes no point save itself, thus it is the required atom, and we are through. □

Thus transitivity plus atomicity is a first order condition definable in a purely Priorean language that is not purely nominal definable: as far as expressing $\mathbf{L}_0$ conditions is concerned the two sorts overlap in expressive power.

3 The Minimal Logic

The minimal nominal tense logic can be axiomatized by the addition of either of two schemas to K_t, the usual axiomatisation of the minimal Priorean tense logic. The schemas are called the NOM and SWEEP schemas, and to present them we need a little notation. Let an *existential tense* be any unbroken sequence of Ps and Fs. The sequence may contain both Ps and Fs, and we regard the null sequence as an existential tense. Thus $FPFP$ and PPP are existential tenses; PGP *isn't* because it contains an occurrence of the universal operator G. We use E, E', and so on as metavariables across existential tenses. By a *universal tense* is meant any unbroken, possibly mixed, sequence of Gs and Hs, including the null sequence; A, A', and so on are used as metavariables over universal tenses. In the following two schemas, n is a metavariable across nominals, and ϕ and ψ are metavariables across arbitrary wffs.

$$E(n \wedge \phi) \wedge E'(n \wedge \psi) \rightarrow E(n \wedge \phi \wedge \psi) \qquad \text{(NOM).}$$
$$E(n \wedge \phi) \rightarrow A(n \rightarrow \phi) \qquad \text{(SWEEP).}$$

Let's instantiate the NOM schema in i and consider what it says:

$$E(i \wedge \phi) \wedge E'(i \wedge \psi) \rightarrow E(i \wedge \phi \wedge \psi).$$

Think of the points of a model as boxes holding items of information. Suppose we are standing at a point t in some model $\mathbf{M}$ and we know that both $E(i \wedge \phi)$ and $E'(i \wedge \psi)$ are true. This means we know that if we follow a certain zig-zag path from t, (the one coded up by E), we can get to a box marked i and containing the information ϕ; and that if we follow another possibly different path from t, (the one coded up by E') we get to another box, also marked i, and containing the information ψ. *But there is only one box marked i.* Hence this single box contains both the information ϕ and the information ψ, and the paths coded for by E and E' lead to the same point. This is precisely what the consequent of NOM gives us. In a nutshell, the NOM schema consists of all the path equations that must be satisfied in any model.

Let K_{nt} be the axiomatisation obtained by adjoining to K_t either of these schemas. We wish to show that K_{nt} captures the minimal logic for NTL. The soundness of either schema is immediate (it's essentially the above 'box and path' argument). Moreover as we will now see, completeness follows straightforwardly by a Henkin argument. In what follows we assume the usual definitions of such concepts as *consistency* and *maximal consistent sets of sentences* (MCS), and all the usual tense logical lemmas and definitions used in Henkin proofs; consult [6] for example. Note that Lindenbaum's Lemma holds. We assume that the set of wffs of our language has been ordered; by Σ^∞ we mean the Lindenbaum expansion of a consistent set of sentences Σ with respect to this ordering.

To build our models we use generated subframes of the canonical Henkin frame $\mathbf{H}^{\mathrm{K}_{nt}}$. (The canonical Henkin frame for K_{nt} is the frame $\mathbf{H}^{\mathrm{K}_{nt}} = \langle H, <_h \rangle$, where H consists of all and only the K_{nt} MCSs; and for all h, $h' \in H$, $h <_h h'$ iff for all wffs ϕ, $\phi \in h'$ implies $F\phi \in h$.) But why use generated subframes of $\mathbf{H}^{\mathrm{K}_{nt}}$? Why not build the usual 'canonical model' using the whole of $\mathbf{H}^{\mathrm{K}_{nt}}$ and the 'natural valuation'? In fact we *cannot* do this: the 'natural mapping' V from the atoms of our language to H defined by $V(a) = \{h \in H : a \in h\}$ is *not* a valuation as each nominal occurs in more than one point of H. (To see this note that for any nominal i both $\{i \wedge p\}$ and $\{i \wedge \neg p\}$ are consistent sets of sentences. Thus $\{i \wedge p\}^\infty$ and $\{i \wedge \neg p\}^\infty$ are distinct points in the Henkin frame that contain i.) By restricting ourselves to generated subframes of $\mathbf{H}^{\mathrm{K}_{nt}}$, however, we will be able to build a valuation from the natural mapping. So, given a consistent set of sentences Σ, take the subframe of $\mathbf{H}^{\mathrm{K}_{nt}}$ generated by Σ^∞. The key lemma is:

Lemma 3.1 (Unique Occurrence Lemma) Let $\mathbf{H}^\Sigma = \langle H^\Sigma, <_h \rangle$ be the subframe of $\mathbf{H}^{\mathrm{K}_{nt}}$ generated by Σ^∞. Then for all $h, h' \in H^\Sigma$, and every nominal i, if $i \in h$ and $i \in h'$ then $h = h'$.

Proof:

Suppose there are two distinct points h, $h' \in H^\Sigma$ that contain the same nominal i. As they are distinct MCS there is some wff ϕ that distinguishes them, so suppose $i \wedge \phi \in h$ and $i \wedge \neg\phi \in h'$. Now as $\mathbf{H}^\Sigma$ is generated from Σ^∞, there is a path from Σ^∞ to h, and a path from Σ^∞ to h'. By appeal to tense logical lemmas we can thus show that there are existential tenses E and E' such that both $E(i \wedge \phi)$ and $E'(i \wedge \neg\phi) \in \Sigma^\infty$. But by NOM this means that $E(i \wedge \phi \wedge \neg\phi) \in \Sigma^\infty$, and thus, by tense logic, we have $E(\phi \wedge \neg\phi) \in \Sigma^\infty$. But this is impossible as Σ^∞ is consistent. □

Now it can happen that not all nominals of our language appear in some $h \in H^\Sigma$. For example, for any choice of i the (consistent) set of sentences $\Sigma = \{\neg Ei : E \text{ is an existential tense}\}$ 'forces i out' of the subframe generated by Σ^∞. But this is easy to fix. Simply adjoin a new point h^+ to $\mathbf{H}^\Sigma$ that is unrelated to any other point, and define a new mapping V_n^+ that is identical to V_n, save only that where V_n assigns $\emptyset$ to some nominal i, V_N^+ assigns $\{h^+\}$ to the same nominal. Clearly V_n^+ is a valuation. The usual induction then shows that $\langle \mathbf{H}^\Sigma, V_n^+ \rangle \models \Sigma[\Sigma^\infty]$ and we have our completeness result.

This method of taking a generated subframe of the Henkin frame, and adding an extra point if necessary, underlies the extended completeness results of the following section. We call the models formed in this way 'Henkin models', and refer to the procedure sketched above as 'generating a Henkin model'.

It is clear that this method of proof also yields a completeness result for languages of nominal modal logic. The modal analogs of existential and universal tenses are unbroken (possibly null) sequences of $\Diamond$s, and of $\Box$s respectively. With the E and A metavariables read in this fashion we have that either K+NOM or K+SWEEP axiomatises the minimal nominal modal logic, where K is the usual axiomatisation of minimal normal modal logic. We refer to either axiomatisation as K_{nm}.

However let us re-examine the proof of the Unique Occurrence Lemma; a little reflection shows that we can do better. In the above proof we made use of three distinct points (h, h' and Σ^∞) and two different paths. But we could have just used a 'two point argument':

given h and h' as described above there must be a path from one to the other — we needn't bring the generating point Σ^∞ explicitly into the proof at all. But once this is observed it becomes clear that we don't need all the instances of either NOM or SWEEP to guarantee completeness; the instances of the following two weakened forms will suffice:

$$\begin{array}{ll} n \wedge E(n \wedge \phi) \to \phi & (\mathrm{NOM}_w) \\ (n \wedge \phi) \to A(n \to \phi) & (\mathrm{SWEEP}_w) \end{array}$$

To see this, we sketch a proof of a Unique Occurrence Lemma from the new axiomatic bases. We treat the case for SWEEP_w. Let our assumptions and notation be as before. Suppose two points h and h' in $\mathbf{H}^\Sigma$ contain the same nominal i. As $\mathbf{H}^\Sigma$ is generated from a single point Σ^∞ it is connected, and thus there is a path between h and h'. Let $A^{(h \to h')}$ be the universal tense that corresponds to the path as seen from h. (That is, starting at h we traverse the path until we reach h', writing down a G for every move forward in time, and H for every move backwards.) All instances of SWEEP_w occur in h, so in particular we have that

$$i \wedge \phi \to A^{(h \to h')}(i \to \phi) \ \in h.$$

But as $i \in h$, then for all $\phi \in h$ we have that $A^{(h \to h')}(i \to \phi) \ \in h$. But then by the usual tense logical lemmas we have that $i \to \phi \in h'$, and as $i \in h'$ we have that $\phi \in h'$. As h and h' are MCS this means that $h = h'$. Thus we have an improved completeness result.

However note that this improvement does *not* hold for modal languages. Intuitively, we have to use a 'three point argument' in modal languages as in such languages we can never look back. The 'two point argument' is the perogative of tense logic. It is straightforward to turn this intuition into a proof that neither $\mathrm{K} + \mathrm{NOM}_w$ nor $\mathrm{K} + \mathrm{SWEEP}_w$ suffices to axiomatise the minimal nominal modal logic. We will proceed by finding a semantical property which distinguishes the derivable from the non-derivable wffs. The first step is to define:

Definition 3.1 Let $\mathbf{T}$ be a frame and t and t' be distinct elements of T. We say t and t' are a *separated pair* iff there is no modal path from t to t', and no modal path from t' to t. A frame is said to *separated* iff it contains at least one separated pair. □

(Note that we talked of *modal* paths, not zig-zag paths, in the above definition.) We now change the interpretation of modal languages with nominals. Let $\mathcal{L}$ be any language of nominal modal logic. In the *separated interpretation* for $\mathcal{L}$ we define *separated valuations* on separated frames; in each separated valuation every nominal denotes exactly two distinct points, t and t', where t and t' are a separated pair. Everything else is as usual: variables denote arbitrary subsets of such frames and the non-atomic sentences are evaluated as usual. We say that an $\mathcal{L}$-wff ϕ is s-valid iff it is valid in any separated interpretation on any separated frame. Clearly both $\mathrm{K} + \mathrm{NOM}_w$ and $\mathrm{K} + \mathrm{SWEEP}_w$ are sound with respect to this interpretation; everything provable from either basis is s-valid. However it is easy to falsify instances of both NOM and SWEEP. Let $\mathbf{T}$ be the frame $\langle \{-1, 0, 1\}, \{\langle 0, -1\rangle, \langle 0, 1\rangle\}\rangle$. Clearly -1 and 1 are a separated pair. Let V be any valuation that assigns $\{-1, 1\}$ to i, and $\{1\}$ to p. Then both an instance of NOM, namely $\Diamond(i \wedge p) \wedge \Diamond(i \wedge \neg p) \to \Diamond(i \wedge p \wedge \neg p)$, and an instance of SWEEP, namely $\Diamond(i \wedge p) \to \Box(i \to p)$, are false at 0 and thus cannot be derived from the weakened basis.

In passing, there's some simple observations we can make about the impact the addition of nominals has on the Henkin frame of the minimal normal modal logic. Suppose we are working with a standard language of modal logic; that is, we have variables and no nominals. Let K be the usual minimal modal logic axiomatisation and let $\mathbf{H}^{\mathrm{K}}$ its canonical frame. The following facts about $\mathbf{H}^{\mathrm{K}}$ are well known: $\mathbf{H}^{\mathrm{K}}$ is left directed, point generated, and indeed *strongly generated*. By this last is meant that there exists an $h \in H^{\mathrm{K}}$ such that for all $h' \in H^{\mathrm{K}}$, $h <_h h'$; from h we can get to any other point in one step. These properties follow from the the fact that K admits the *Law of Disjunction* (LOD): $\vdash_L \Box\phi_1 \vee \ldots \vee \Box\phi_n$ implies $\vdash \phi_m$, for some m such that $1 \leq m \leq n$. For a discussion of why these properties follow from LOD see [14].

The minimal nominal modal logic, however, does not admit LOD. Note that $\mathbf{H}^{\mathrm{K}_{nm}}$ cannot be left directed as no MCS h can precede both $\{i \wedge \phi\}^\infty$ and $\{i \wedge \neg\phi\}^\infty$; hence LOD cannot hold. This example also shows that $\mathbf{H}^{K_{nm}}$ cannot be strongly generated. In fact, it can't even be generated: for arbitrary existential modalities $\Diamond^n$ and $\Diamond^m$, $\Diamond^n(i \wedge \phi) \wedge \Diamond^m(i \wedge \neg\phi)$ is inconsistent, and thus no MCS h can precede both $\{i \wedge \phi\}^\infty$ and $\{i \wedge \neg\phi\}^\infty$, no matter how many steps intervene. The only obvious thing we can say about the structure of $\mathbf{H}^{K_{nm}}$ derives from the following observation: one special case of LOD is unaffected by the addition of nominals: $\vdash_{K_{nm}} \phi$ iff $\vdash_{K_{nm}} \Box\phi$, and thus $\mathbf{H}^{K_{nm}}$ is left unbounded.

As was mentioned in the introduction, a group of Bulgarian logicians have considered modal languages enriched by nominals. Let us consider the axiomatisation of the minimal modal logic for languages with nominals due to Gargov, Passy and Tinchev [10]. They first define necessity and possibility forms:

Definition 3.2 Let $\mathcal{L}$ be a language of NML, \$ be a new entity distinct from any $\mathcal{L}$ wff or symbol, and θ be a wff of $\mathcal{L}$. Then the *necessity forms* of $\mathcal{L}$, are the elements of the smallest set $\Box$-form such that:

$$\begin{array}{l} \$ \in \Box\text{-form} \\ L \in \Box\text{-form implies } \theta \rightarrow L \in \Box\text{-form} \\ L \in \Box\text{-form implies } \Box L \in \Box\text{-form;} \end{array}$$

and the *possibility forms* of $\mathcal{L}$, are the elements of the smallest set $\Diamond$-form such that:

$$\begin{array}{l} \$ \in \Diamond\text{-form} \\ L \in \Diamond\text{-form implies } \theta \wedge L \in \Diamond\text{-form} \\ L \in \Diamond\text{-form implies } \Diamond L \in \Diamond\text{-form.} \end{array}$$

If ψ is any wff of $\mathcal{L}$, and L and M are $\Box$-forms and $\Diamond$-forms respectively, then by $L(\psi)$ and $M(\psi)$ are meant the $\mathcal{L}$-wffs obtained by replacing the (unique) occurrence of \$ in L and M respectively by ψ. □

They then axiomatise the minimal logic for languages of weak NML by adding to the usual axioms of the minimal modal logic K all instances of the following schema:

$$M(n \wedge \phi) \rightarrow L(n \rightarrow \phi) \qquad (\mathrm{Ax}_N)$$

where L and M are metavariables over $\Box$-forms and $\Diamond$-forms respectively. They prove completeness by a three point argument on generated subframes of the $\mathbf{H}^{\mathrm{K}_{nm}}$. The form of this schema is superficially reminiscent of that of SWEEP, but the M and the L don't range over universal and existential modalities but over the more complex $\Box-$ and $\Diamond-$ forms. Thus for fixed i and ϕ the consequents of Ax_N include all entries in the following matrix:

$$\begin{array}{llll}
(i \to \phi) & \Box(i \to \phi) & \Box\Box(i \to \phi) & \cdots \\
\phi \to (i \to \phi) & \Box(\phi \to (i \to \phi)) & \Box\Box(\phi \to (i \to \phi)) & \cdots \\
\phi \to (\phi \to (i \to \phi)) & \Box(\phi \to (\phi \to (i \to \phi))) & \Box\Box(\phi \to (\phi \to (i \to \phi))) & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
\phi \to \Box(i \to \phi) & \Box(\phi \to \Box(i \to \phi)) & \Box\Box(\phi \to \Box(i \to \phi)) & \cdots \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

The antecedents of Ax_N, again for fixed i and ϕ, consists of all entries in a similar matrix obtained from that above by replacing $\Box$ by $\Diamond$ and $\to$ by $\wedge$. Note that for fixed i and ϕ the SWEEP schema contains only conditionals formed from the first row of each of these two matrices. The simpler SWEEP$_w$ schema that suffices for tense logic essentially consists, for fixed i and ϕ, of only the single wff $i \wedge \phi$ as antecedent; and as consequents just the wffs in the first row of the above matrix.

We conclude this section by noting some theorems of the minimal tense logic. Firstly, nominals interact strongly with universal tenses; Hi and Gi can only be true under 'end conditions', hence both the following 'end effects' $i \wedge Gi \wedge F\phi \to \phi$ and $Gi \wedge F\psi \to G\psi$ are theorems. Note that if we replace i by p in the above the resulting wffs are *not* Priorean valid. Next, suppose t is a point and that there is a path that leads away from t but eventually returns there. Then a 'reverse journey' exists: we could traverse the path in the reverse direction and still get back to t. In NTL we can talk about such reverse journeys; we can't in standard languages as we cannot uniquely mark the starting point. To display the relevant theorem we first need to define the ***transposition*** E^T of an existential tense E. By this is meant the existential tense formed by reversing the sequence of tenses in E and forming the mirror image. For example, $(PPFPF)^T = PFPFF$. If an existential tense E codes a path between points t and t' as seen by an observer at t, then E^T codes the same path as viewed by an observer at t'. The theorem asserting the existence of reverse journeys is: $i \wedge Ei \to E^T i$. Again note that if we replace i by p we do not get a Priorean validity. Finally note that if we can break off a journey in the middle, pick up a piece of data, and then continue round, we can do the same thing backwards: $i \wedge E_1(\psi \wedge E_2 i) \to E_2^T(\psi \wedge E_1^T i)$.

4 Extensions of K_{nt}

In this section we obtain completeness results for some classes of frames of temporal interest. The major technical point of this section is that once two simple results have been noted — the Irreflexivity Lemma and the Antisymmetry Lemma — Segerberg's [25] cluster manipulation techniques can be straightforwardly applied. We illustrate this by considering the logics of four classes of frames of temporal interest: the SPOs, STOs, POs and TOs. The proofs presented here make use only of standard modal techniques: no use is made of additional rules of inference. We briefly discuss the use of such rules and raise a general question concerning their eliminability. We then turn to algebraic semantics and show that a Thomason style [28] adequacy theorem can be proved for NTL.

Languages of NTL inherit a number of completeness results straightforwardly from Priorean tense logic. For example, adding to K_{nt} as axioms all instances of $FF\phi \to F\phi$ (*4*), $\phi \to F\phi$ (*T*), $P\top \wedge F\top$ (*D*), or the *Lin* schema, which consists of the conjunction of

$$F\phi \wedge F\psi \to F(\phi \wedge F\psi) \vee F(\psi \wedge F\phi) \vee F(\phi \wedge \psi) \qquad (RLin),$$

with its mirror image *LLin*, yields Henkin frames that are transitive, reflexive, both right and left unbounded, and linear respectively; hence generating Henkin models (adding isolated points if necessary) as described in the previous section gives an immediate crop of completeness results.

This is useful, but unsurprising. Moreover these completeness theorems deal only with classes of frames already definable in Priorean languages. How do we axiomatise the newly definable classes, and can the defining formulas be used as axioms? They can, but not in quite so straightforward a fashion as for the examples listed above. Although any instance of $n \rightarrow \neg Fn$ (I) defines irreflexivity, the inclusion of all instances of this schema as axioms does *not* guarantee an irreflexive Henkin frame; and although any instance of $n \rightarrow G(Fn \rightarrow n)$ $(Anti)$ defines antisymmetry, the inclusion of all instances of this schema as axioms does *not* guarantee an antisymmetric Henkin frame. The problem is that not all MCSs in the Henkin frame contain nominals, and we cannot guarantee that points without nominals have the desired property. Crucially, however, points in these Henkin frames that do contain nominals are well behaved. Let's consider what happens in the case of irreflexive extensions of K_{nt}:

Lemma 4.1 (Irreflexivity Lemma) Let A be any axiomatisation extending K_{nt} that contains all instances of the I schema, and let $\mathbf{H}$ be the Henkin frame for A. Then for any point h in $\mathbf{H}$ containing a nominal, $h \not< h$ where $<$ is the usual Henkin ordering.

Proof:

Suppose h is a point in this Henkin frame that contains a nominal, say i. Suppose for the sake of a contradiction that $h < h$. By definition this means that that for all wffs ϕ, $\phi \in h$ implies $F\phi \in h$. In particular, as $i \in h$ this means that $Fi \in h$. But $i \rightarrow \neg Fi$ is an axiom, so $i \rightarrow \neg Fi \in h$. As $i \in h$ this means $\neg Fi \in h$. As h is an MCS we have a contradiction. $\Box$

This result tells us that 'defects' of the Henkin model (here, reflexive points, or more generally *clusters*) are localised to points not containing nominals. This means that if we can find a way of repairing the Henkin model that only acts on the defective points then we don't have to worry about destroying the unique occurrence property enjoyed by the Henkin model. Segerberg's bulldozing technique works in this manner. Let's see how we can apply bulldozing to prove a completeness result for the SPOs. Let I4 be K_{nt} augmented by all instances of *I*, the irreflexivity schema, and *4*, the transitivity schema. As we shall see, I4 axiomatises the nominal tense logic of the SPOs.

First some (standard) terminology. By a *cluster* C of a frame $\langle T, < \rangle$ is meant any non-empty subset C of T such that $(C \times C) \cap <$ is an equivalence relation, and for no proper superset C' of C is $(C' \times C') \cap <$ an equivalence relation. A cluster is *proper* if it contains at least two points, and *simple* otherwise. Clusters are the defects of transitive Henkin frames and the essence of Segerberg's bulldozing technique is to remove all the clusters C, replacing each cluster with the lexicographical product $\mathbf{S} \odot C$ where where $\mathbf{S}$ is some unbounded STO. This procedure produces a new SPOed model $\mathbf{B}$, and as the Henkin model is a p-morphic image of this model they are equivalent. For further details the reader is referred to either Segerberg's original paper [25], or Goldblatt's more recent account [9].

Theorem 4.1 I4 is strongly complete with respect to the class of SPOed frames.

Proof:

Given an I4-consistent set of sentences Σ form Σ^{∞} and generate a Henkin model $\langle \mathbf{T}, V_t \rangle$ for Σ, adding an extra (irreflexive) point if necessary, as described in the last section. Bulldoze $\mathbf{T}$, and choosing (say) $\mathbf{Z}$ to form the lexicographical product. The new structure $\mathbf{B}$ is an NTL model; that is, nominals are assigned singletons in this structure. This follows from the Irreflexivity Lemma: we know that every point t of the Henkin model that contains a nominal is irreflexive, hence no such point is in a cluster and thus no such point was bulldozed. Thus $\mathbf{B}$ is an SPOed model of which the Henkin model is a p-morphic image, and so $\mathbf{B}$ verifies Σ as required. □

Let's next consider the STOs. Let LIN_s be I4 extended with all instances of $Pn \vee n \vee Fn$ (*Tri*) the trichotomy schema, together with all instances of *Lin*, the linearity schema.

Theorem 4.2 LIN_s is strongly complete with respect to the class of STOs.

Proof:
Given a LIN_s consistent set of sentences Σ, form Σ^∞. Let $\langle \mathbf{T}, V\rangle$ be the substructure of the Henkin model generated by Σ^∞. This substructure is already a model, for the natural mapping V assigns every nominal a singleton. That every nominal is assigned at most one point is just the usual unique occurrence argument. However it is also the case that there are no 'unassigned nominals'. We see this as follows. Suppose for the sake of a contradiction that some nominal, say j is not assigned any $t \in T$. But as all instances of the *Tri* schema are axioms, then $Pj \vee j \vee Fj \in \Sigma^\infty$. This tells us that either j is in Σ^∞, or in some other MCS Σ^j such that $\Sigma^j < \Sigma^\infty$ or $\Sigma^\infty < \Sigma^j$. This follows by standard tense logical reasoning from the fact that Σ^∞ is consistent.

Thus $\langle \mathbf{T}, V\rangle$ is a model. Moreover, because of the presence of the transitivity and linearity schemas it is transitive and linear. The only thing that could be wrong with this model is that it contains clusters, but just as in the previous proof we can remove these by bulldozing. As before, the Irreflexivity Lemma guarantees that the unique occurrence property is unaffected. We thus have verified our original set of sentences Σ on a STO and are through. □

These two completeness proofs show the essence of what is involved in applying cluster manipulation techniques to languages with nominals: standard techniques may be used as long as they don't ruin the unique occurrence property. Indeed once this is seen it is clear that the nominal tense logics of $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{N}$ and $\mathbf{R}$ can be axiomatised simply by adding all instances of NOM_w, I and Tr, to their Priorean tense logical axiomatisations. All the real work has been done for us: simply inspecting the proofs given in either [25] or [9] shows that none of the cluster manipulation techniques employed effect unique occurrence. (The reader who would prefer to see more details is referred to [3].)

A second point is worth making. When working on linear frames we don't need all the instances of NOM_w; most of the path equations are redundant because of the simple geometry. Define $[U]\phi$ to be $H\phi \wedge \phi \wedge G\phi$. Over linear frames $[U]$ means 'at all times'. Then instead of all instances of NOM_w we need merely take as axioms all instances of:

$$i \wedge \phi \rightarrow [U](i \rightarrow \phi).$$

Clearly this forces unique occurrence over linear time flows.

Actually, the universal modality $[U]$ is an extremely natural addition to languages with nominals. Note that such wffs as $[U](i \rightarrow \phi)$ in effect shift the point of evaluation to the point named by i, and test ϕ there. Indeed when working over non-linear time flows over which $[U]$ is not definable, it is well worth adding it as an additional primitive operator. That is, it is well worth adding to NTL a new primitive modality $[U]$ such that

$$\mathbf{M} \models [U]\phi[t] \text{ iff } \mathbf{M} \models \phi[t'], \text{ for all points } t'.$$

I won't discuss this idea further here except to mention that every author who has considered nominals done this. The methods of the present section extend to such enriched languages; full details are given in [3]. General results concerning the universal modality may be found in [13].

Let's now look at partially ordered frames. Our investigation will be analogous to that given above for the strictly partially ordered frames. First of all, we must ensure that that the antisymmetry schema $n \to G(Fn \to n)$ guarantees the proper behaviour of those points in the Henkin frame containing nominals. It does, as the following lemma shows:

Lemma 4.2 (Antisymmetry Lemma) Let A be any axiomatisation extending K_{nt} that contains all instances of the *Anti* schema $n \to G(Fn \to n)$, and let $\mathbf{H}$ be the Henkin frame for this axiomatisation. Then for any points h and h' in this Henkin frame such that h contains a nominal, $h < h'$ and $h' < h$ implies $h = h'$, where $<$ is the usual Henkin ordering.

Proof: Suppose h contains a nominal, say i, and further suppose that there is a point h' such that $h < h'$ and $h' < h$. As all instances of *Anti* are axioms, $i \to G(Fi \to i) \in h$, thus as $i \in h$ we have that $G(Fi \to i) \in h$. As $h < h'$, it follows that $Fi \to i \in h'$. But as $h' < h$ and $i \in h$, we have that $Fi \in h'$. Hence $i \in h'$. But as A is an extension of K_{nt}, $\mathbf{H}$ has the unique occurrence property, thus $h = h'$. □

We use this as follows. As we are working with partial orders every point is reflexive, thus every point is in a cluster; what the Antisymmetry Lemma establishes is that no point containing a nominal is in a *proper* cluster. Points containing nominals are simple clusters. Our basic strategy is clear: we will bulldoze $\mathbf{T}$, but bulldoze only *proper* clusters. Also, because we want reflexive models, when we form the lexicographical products $\mathbf{S} \odot C$ used in the bulldozing process, we choose $\mathbf{S}$ to be some infinite TO, such as $\langle Z, \leq \rangle$. Let PO be the axiomatisation obtained by adding as axioms all instances of *4*, *T* (the reflexivity schema) and *Anti* to the axioms of K_{nt}. PO is characterised by the class of all POs. Soundness is obvious. As for completeness:

Theorem 4.3 PO is strongly complete with respect to the class of all partial orders.

Proof:

Given a PO-consistent set of sentences Σ, form Σ^{∞} and generate a Henkin model, adding an extra (reflexive) point if need be. This model is guaranteed to be both reflexive and transitive due to the presence of the *4* and *T* axioms, but antisymmetry is not assured. Bulldoze the proper clusters as just described to form $\mathbf{B}$. By the Antisymmetry Lemma we know that no point containing a nominal belonged to a proper cluster, hence the unique occurrence property was not affected by the bulldozing. Thus $\mathbf{B}$ is a POed model for Σ. □

Finally, let's consider the TOs. Let LIN be PO augmented by all instances of $Pi \vee i \vee Fi$ (*Tri*) and *Lin*. (Actually, as noted above, we don't need all the instances of NOM_w that PO contains; over linear frames all instances of $i \wedge \phi \to [U](i \to \phi)$, where $[U]\phi$ is defined to be $H\phi \wedge \phi \wedge G\phi$, suffices to force unique occurrence.) Then by reasoning analogous to that used in the previous two theorems we establish:

Theorem 4.4 LIN is strongly complete with respect to the TOs. □

The preceding discussion shows that standard modal techniques can be straightforwardly adapted to languages with nominals. However there is another way to proceed, which is explored in the Bulgarian work: to make use of an additional rule of inference. Recall from the previous section the definition of necessity forms L. The rule of inference called COV is defined as follows: for any necessity form L, from $\vdash L(\neg i)$ infer $\vdash L(\bot)$, where i does not occur in the L form. To give a simple instance, let $\neg\phi$ be a wff not containing i. Then from $\vdash \neg\phi \to \neg i$ we can deduce $\vdash \neg\phi \to \bot$; or, simplifying, from $\vdash i \to \phi$ we can infer $\vdash \phi$.

As is shown in [10] and [12], the use of this rule enables one to build Henkin models in which every point contains a nominal. The usefulness of this should be clear. By the Antisymmetry and Irreflexivity Lemmas, for example, this means that we are able to build antisymmetric and irreflexive models in one step.

Such rules have been used before in modal logic, and with similar motivations. Their first use seems to have been due to Gabbay [7], who introduced them precisely to enable irreflexive models to be directly built. More recently they have been used in other modal logics for similar purposes. Gabbay and Hodkinson [8] use them in Until-Since logic; de Rijke [23] in D-logic; and finally Venema proves a number of interesting results about their use. None of these last mentioned papers makes use of nominals, but the way the rules the rules are used to label nodes with fresh propositional variables is very similar.

It seems likely that the use of such rules will become a standard tool in modal logic; as the above list shows, they are highly adaptable, and when working with more complex intensional languages can greatly simplify competeness proofs. But one topic doesn't seem to have been addressed: when are these rules *necessary*? As the results of this section show, when working with Priorean Languages enriched with nominals, such rules aren't actually needed to axiomatise many natural classes of frames. That is, the above completeness result can be viewed as COV elimination results. It would be very useful to obtain a characterisation of when such rules are eliminable; in the meantime, the results of this section show that in many important cases elimination is possible.

To close this section we shall show how to define an algebraic semantics for NTL and prove its adequacy in the sense of Thomason [28]. In order to prove this we must first be precise about what a nominal tense logic is. We define:

Definition 4.1 A *nominal tense logic* is a set of wffs that contains all the K_{nt} axioms and is closed under modus ponens, temporal generalisation, and NTL substitution. □

This differs from the standard definition of a tense logic only in not demanding closure under unrestricted substitutions. Note that the theory of any class of frames is a nominal tense logic.

Let $\mathcal{B} = \langle B, 0, 1, -, +, \times, p, f\rangle$ be a *temporal algebra* as defined by Thomason. That is, $\langle B, 0, 1, -, +, \times\rangle$ is a Boolean algebra, where B is the carrier set, whose elements are represented by $b, b', b'', \ldots$; 0 is bottom; 1 top; $-$, $+$ and $\times$ have the obvious reading and p and f are unary operators on B such that $f(0) = p(0) = 0$; $f(b + b') = f(b) + f(b')$; $p(b + b') = p(b) + p(b')$; and $f(b) \times b' = 0$ iff $b \times p(b') = 0$. Any Priorean wff corresponds in the obvious way to a polynomial over algebras of this signature, and from the work of Thomason we know that the class of temporal algebras provides an adequate semantics for Priorean tense logic.

How can we adapt this semantics to NTL? A plausible (but misguided) attempt is as follows. Define an *atomic temporal algebra* to be a temporal algebra whose underlying Boolean algebra is atomic. Interpet wffs of NTL on atomic temporal algebras by insisting that nominals denote atoms. It is an easy exercise to see that this interpretation of NTL is sound. Unfortunately we don't get much further than this: when we come to prove adequacy we run into a problem as the Lindenbaum algebra is not atomic.

The way to proceed is as follows. Define a *nominal temporal algebra* to be a pair $\mathcal{N} = \langle \mathcal{B}, I\rangle$ where $\mathcal{B}$ is a temporal algebra and I a non-empty subset of B such that for all $i \in I$, for all $b \in B$ and for all compositions e of the f and p operators (including the null composition), $i \times e(i \times b) \times -b = 0$, the algebraic analog of the NOM$_w$ schema. Note that any atomic temporal algebra is a nominal temporal algebra: in this case the non-empty subset $\mathcal{B}$ is the set of atoms. However there are many other nominal temporal algebras, and (as we shall see) this includes the Lindenbaum algebras.

Given such an algebra we interpret a language of NTL on it in the obvious fashion. That is, each wff of NTL corresponds to a polynomial as before, but we insist that the variables in these polynomials that correspond to nominals range over all and only the elements of I. In short, we have moved to a two sorted algebraic semantics — the sorts in question being B and I — and when evaluating our polynomials we only consider the sortally correct evaluation possibilities. We say that a nominal temporal algebra $\mathcal{B}$ validates ϕ ($\mathcal{B} \models \phi$) iff $h_\phi = 1$ identically in $\mathcal{B}$; and we say that ϕ is algebraically valid iff $\mathcal{B} \models \phi$ for all nominal temporal algebras $\mathcal{B}$.

Given any algebra $\mathcal{N} = \langle \mathcal{B}, I \rangle$ of the same type as a nominals temporal algebra — that is, $\mathcal{B}$ is an algebra of the same type as a temporal algebra, and I is a non-empty subset of $\mathcal{B}$'s carrier set B — we can interpret a wff ϕ on $\mathcal{N}$ exactly as though $\mathcal{N}$ were a nominal temporal algebra. However, following Thomason, we have that the nominal temporal algebras are precisely the algebras that validate the K_{nt} theorems:

Lemma 4.3 Let $\mathcal{N} = \langle \mathcal{B}, I \rangle$ be an algebra of the type of nominal temporal algebras. Then $\mathcal{N}$ validates all theorems of K_{nt} iff $\mathcal{N}$ is a nominal temporal algebra.

Proof:

Suppose $\langle \mathbf{B}, I \rangle$ is a model for K_{nt}. As K_{nt} contains minimal Priorean tense logic it follows by Thomason's proof that $\mathcal{B}$ must be a temporal algebra. Only the part peculiar to the nominals thus remains, and this is immediate: if $\langle \mathcal{B}, I \rangle$ is a model for K_{nt} it validates all instances of $i \wedge E(i \wedge \phi) \to \phi$, that is, $\neg(i \wedge E(i \wedge \phi) \to \phi) \leftrightarrow \bot$, which means that in $\langle \mathcal{B}, I \rangle$, $i \times e(i \times b) \times -b = 0$, for all $i \in I$, $b \in B$, and composition sequences e.

Proving that any nominal temporal algebra is a model for K_{nt} merely involves the usual inductive soundness argument. □

This shows that the set of all algebraically valid NTL wffs is identical to the set of wffs valid on all frames. Now we are ready for the adequacy result. We will be able to build the required algebras using the Lindenbaum construction.

Theorem 4.5 (Adequacy Theorem) For any nominal tense logic L there is nominal tense algebra $\mathcal{N}^L = \langle \mathcal{B}, I \rangle$ such that $\mathcal{N}^L \models \phi$ iff $\vdash_L \phi$.

Proof:

Let $\mathcal{N}$ be the Lindenbaum algebra for L. That is, $B = \{[\phi] : \phi \in WFF\}$; $[\phi] = [\psi]$ iff $\vdash_L \phi \leftrightarrow \psi$; $0 = [\bot]$; $1 = [\top]$; $-[\phi] = [\neg\phi]$; $[\phi]+[\psi] = [\phi \vee \psi]$; $[\phi] \times [\psi] = [\phi \wedge \psi]$; $f([\phi]) = [F\phi]$; $p([\phi]) = [P\phi]$; and $I = \{[i] : i \in NOM\}$. By the usual reasoning $\mathcal{B} = \langle B, 0, 1, -, +, \times, f, p \rangle$ must be a temporal algebra. Moreover, as K_{nt} includes among its theorems all instances of NOM$_w$ and $\top$, we have that any instance of NOM$_w$ is equivalent to $\top$ in K_{nt}, and hence in L. This means that for all $[i] \in I$, $[\phi] \in B$ and composition sequences e we have in the Lindenbaum algebra that $-([i] \times e([i] \times [\phi])) + [\phi] = 1$, or $[i] \times e([i] \times [\phi]) \times -[\phi] = 0$. Thus the Lindenbaum algebra is a nominal temporal algebra and hence validates all theorems of K_{nt}. The key point, however, is to show that the Lindenbaum algebra validates not just the K_{nt} theorems, but all L theorems, and here is where we use the fact that nominal tense logics are closed under NTL substitution. Let ϕ be any wff. When we evaluate the polynomial h_ϕ on the Lindenbaum algebra, we obtain a value $[S(\phi)]$. But it follows by induction on $deg(\phi)$

that $S(\phi)$ is an NTL substitution instance of ϕ. So suppose $\vdash_L \phi$. As L is closed under NTL substitutions, $\vdash_L S(\phi)$ for all NTL substitution instances $S(\phi)$ of ϕ. Hence for all such $S(\phi)$, $[S(\phi)] = 1$, thus $h_\phi = 1$ identically, and L's Lindenbaum algebra validates every L theorem.

To show that any non-theorem ϕ of L is falsified in this algebra, suppose that $\not\vdash_L \phi$. If for each variable p and nominal i in ϕ we evaluate h_ϕ with the corresponding polynomial variables interpreted by $[p]$ and $[i]$ respectively, we obtain $[\phi]$. Clearly $[\phi] \neq [1]$ as otherwise $\vdash_L \phi$, which contradicts our assumption of ϕ's non-theoremhood. □

5 Decidability and the finite model property

The logics we have considered so far are decidable. This is unsurprising: what is interesting is that in spite of an apparent obstacle filtration methods can be used to prove it.

In Priorean tense logic filtrations provide a reasonably general method for establishing decidability. A typical proof runs as follows. Given an axiomatisation $\mathrm{K}_t\mathrm{S}$ which we know to be complete with respect to some class of frames $\mathcal{T}$, we attempt to show that it is also characterised by the class of all finite frames in $\mathcal{T}$. When this can be shown we say that $\mathrm{K}_t\mathrm{S}$ has the *finite frame property* with respect to $\mathcal{T}$. The finite frame property is commonly established by using filtrations. Given that the theorems of $\mathrm{K}_t\mathrm{S}$ form an r.e. set (and in most cases of interest they will) this usually establishes decidability. For, if the finite frames in $\mathcal{T}$ form an r.e. set (and again, in most cases of interest they will) searching through all the finite models based on these frames is an effective (if inefficient) procedure for generating all the non-theorems of $\mathrm{K}_t\mathrm{S}$, and thus the set of $\mathrm{K}_t\mathrm{S}$ theorems is recursive.

Matters are not quite so straightforward in NTL as many obvious axiomatisations lack the finite frame property. This is illustrated by the axiomatisation I4D. This is I4 augmented by the D axiom $P\top \wedge F\top$. This axiom forces the Henkin model to be unbounded in both directions, thus it follows from the methods of the last section that I4D is complete with respect to the class of unbounded STOs. However it is also clear that only infinite frames can validate all the I4D axioms, thus I4D does not have the finite frame property. The way appears blocked — but there is a loophole. Although I4D does not have the finite frame property it does have the *finite model property*. That is, it is possible to define a class of finite models $\mathcal{M}$ such that $\vdash_{\mathrm{I4D}} \phi$ iff $\mathbf{M} \models \phi$, for all $\mathbf{M} \in \mathcal{M}$. The class of models needed will shortly be described, for now merely note that the loophole we are exploiting does not exist in purely Priorean languages: a well known theorem of Segerberg's states that if L is any classical modal logic, then L has the finite model property iff L has the finite frame property [26, page 33]. Thus Segerberg's theorem does not hold in NTL as I4D is a counterexample.

Let's first treat the irreflexive logics of the previous section. Call $\mathbf{T} = \langle T, < \rangle$ an *irreflexivity containing* frame iff there is a $t \in T$ such that $t \not< t$. Call a valuation V on such a frame *irreflexivity respecting* iff for all nominals i, $t \in V(i)$ implies $t \not< t$. That is, irreflexivity respecting valuations send all nominals to irreflexive points. We call $\mathbf{M} = \langle \mathbf{T}, V \rangle$ an I_1 *model* iff $\mathbf{T}$ is an irreflexivity containing frame and V an irreflexivity respecting valuation on $\mathbf{T}$. The class of all I_1 models is called $\mathcal{M}(I_1)$.

Lemma 5.1 I4 is sound and complete with respect to the class of all transitive I_1 models. That is, $\vdash_{\mathrm{I4}} \phi$ iff $\mathbf{M} \models \phi$, for all $\mathbf{M} \in \mathcal{M}(I_1) \cap \mathcal{M}(Tran)$.

Proof:

(Soundness). The only axiom schemas that require checking are 4 and I, the others being universally valid. As the models $\mathbf{M}$ we are considering are transitive, any instance of 4 is true in all such $\mathbf{M}$; and as all nominals denote irreflexive points in $\mathbf{M}$, all instances of I are true in these $\mathbf{M}$. All three rules of inference preserve validity in a model, and so I4 is sound with respect to $\mathcal{M}(I_1) \cap \mathcal{M}(Tran)$.

(Completeness). This is shown by the first part of the completeness proof for I4 given in the previous section; that is, the stage preceding bulldozing. Given any I4-consistent sentence ϕ, the transitive Henkin model we generate verifies ϕ at some point. Moreover the Irreflexivity Lemma shows that every point containing a nominal is irreflexive, and thus this Henkin model is in $\mathcal{M}(I_1) \cap \mathcal{M}(Tran)$. □

However we *don't* need all the models in $\mathcal{M}(I_1) \cap \mathcal{M}(Tran)$ to establish completeness; simply the finite ones will do.

Theorem 5.1 I4 has the finite model property with respect to $\mathcal{M}(I_1) \cap \mathcal{M}(Tran)$. That is, $\vdash_{\mathrm{I4}} \phi$ iff $\mathbf{M} \models \phi$ for all finite $\mathbf{M} \in \mathcal{M}(I_1) \cap \mathcal{M}(Tran)$.

Proof:

Soundness follows from the previous lemma. The following filtration argument establishes completeness. By the previous lemma we know that given an I4-consistent sentence ϕ we can find an $\mathbf{M} \in \mathcal{M}(I_1) \cap \mathcal{M}(Tran)$ such that $\mathbf{M} \models \phi[t]$, at some point t. Now, if ϕ contains occurrences of nominals, define Σ^- to be

$$\{\phi\} \cup \{i \rightarrow \neg Fi : i \text{ occurs in } \phi\};$$

while if ϕ contains no occurrences of nominals choose any nominal (say i) and define Σ^- to be $\{\phi\} \cup \{i \rightarrow \neg Fi\}$. Let Σ be the smallest set of wffs containing Σ^- that is closed under subformulas. Form any Prior filtration $\mathbf{M}^f$ of $\mathbf{M}$ through Σ such that for all nominals $j \notin \Sigma$, $V_f(j) = V_f(i)$, for some nominal $i \in \Sigma$. (By our definition of Σ^- there will always be at least one such i.) By the Filtration Theorem $\mathbf{M}^f \models \phi[E(t)]$. But $\mathbf{M}^f$ is a model in the required class: clearly it is finite, because Σ is a finite set of sentences; and it is transitive because we took a Prior filtration. Moreover $\mathbf{M}^f$ does contain irreflexive points, and all nominals are assigned irreflexive points in this filtration. To see this, note that it follows from the definition of Prior filtrations that:

$$\exists\phi(F\phi \in \Sigma \;\&\; \mathbf{M} \models \phi[t] \;\&\; \mathbf{M} \not\models F\phi[t]) \text{ implies } E(t) \not<_f E(t).$$

But for all nominals $i \in \Sigma$ (and there is at least one) we have that $Fi \in \Sigma$. Further, as our original model was in $\mathcal{M}(I_1)$, $\mathbf{M} \models i[t]$ means $\mathbf{M} \not\models Fi[t]$, and thus for all such points t, $E(t) \not< E(t)$. This means that all points in the filtration $\mathbf{M}^f$ denoted by nominals are irreflexive, and we have our result. □

Thus the familiar 'search through finite structures' argument leads to the desired result:

Corollary 5.1 I4 is decidable. □

Decidability for other extensions of I4 follow from this basic result. For example, I4D is decidable because, as with I4, the (unbulldozed) Henkin model establishes that $\vdash_{\mathrm{I4}} \phi$ iff $\mathbf{M} \models \phi$, for all unbounded $\mathbf{M} \in \mathcal{M}(I_1) \cap \mathcal{M}(Tran)$. As filtrations inherit unboundedness, the filtration given above establishes the finite model property for I4D with respect to this class of models and decidability follows. Similarly, a result for LIN$_s$ is obtained by noting that once again the unbulldozed Henkin model establishes completeness with respect to the class of all trichotomous $\mathbf{M} \in \mathcal{M}(I_1) \cap \mathcal{M}(Tran)$, and as filtrations inherit trichotomy we again have the finite model property and decidability. So we conclude:

Corollary 5.2 I4D and LIN_s are both decidable. $\Box$

Thus we have a tool that works for some of the nominal tense logics of interest above I4. The method doesn't work for the logics of **N**, **Z** or **R**, as these logics are not sound on the classes of finite model produced by this method. However even for Priorean tense logic more powerful techniques are needed in these cases, notably the use of Rabin-Gabbay techniques (see [6] for a survey of these). The Rabin-Gabbay method works equally well for languages with nominals, though I will not discuss this here.

I'll now describe how to prove analogous results for logics of partially ordered frames, that is, for extensions of PO. The concepts we need are essentially those given above but with talk of 'simple clusters' replacing talk of 'irreflexive points'. That is, we define notions of *simple cluster containing* frames and *simple cluster respecting* valuations analogous to those given above. An SC_1 model is a model whose frame is simple cluster containing and whose valuation is simple cluster respecting. We denote the class of all SC_1 models by $\mathcal{M}(SC_1)$. Our usual method of generating Henkin models establishes that $\vdash_{\mathrm{PO}} \phi$ iff $\mathbf{M} \models \phi$, for all $\mathbf{M} \in \mathcal{M}(SC_1) \cap \mathcal{M}(Pre)$, where $\mathcal{M}(Pre)$ is the class of all preordered models; the key point to observe here is that the Antisymmetry Lemma guarantees that every point in the Henkin model denotes a simple cluster. Now we must make the finite models:

Theorem 5.2 PO has the finite model property with respect to $\mathcal{M}(SC_1) \cap \mathcal{M}(Pre)$. That is, $\vdash_{\mathrm{PO}} \phi$ iff $\mathbf{M} \models \phi$ for all finite $\mathbf{M} \in \mathcal{M}(SC_1) \cap \mathcal{M}(Pre)$.

Proof:

Soundness is immediate. To prove completeness we use a filtration argument. Given a PO-consistent wff ϕ containing occurrences of nominals, define Σ^- to be

$$\{\phi\} \cup \{i \rightarrow G(Fi \rightarrow i) : i \text{ occurs in } \phi\},$$

and if ϕ is purely Priorean define Σ^- to be $\{\phi\} \cup \{i \rightarrow G(Fi \rightarrow i)\}$, for some selected nominal i. Let Σ be the smallest set of wffs containing Σ^- that is closed under subformulas.

Given a model $\mathbf{M}$ in $\mathcal{M}(SC_1) \cap \mathcal{M}(Pre)$ that verifies ϕ (and one must exist by the completeness result just noted) Prior filtrate $\mathbf{M}$ through Σ to form $\mathbf{M}^f$. By the Filtration Theorem this model also verifies ϕ. Now $\mathbf{M}^f$ is finite and transitive, and as filtrations inherit reflexivity we have that $\mathbf{M}$ is a finite preorder.

Now we know from our discussion of filtrations that for any nominal i in Σ, if $V^f(i) = \{E(t)\}$ then t is in fact the unique element of $V(i)$. Suppose that there is a point $E(t')$ in the filtration such that $E(t) <_f E(t')$ and $E(t') <_f E(t)$. As $\mathbf{M} \models G(Fi \rightarrow i)[t]$ (which it does, because in the original model both i and $i \rightarrow G(Fi \rightarrow i)$ are true at t), and as $G(Fi \rightarrow i) \in \Sigma$, $\mathbf{M}^f \models G(Fi \rightarrow i)[E(t)]$ by the Filtration Theorem. But as $E(t) <_f E(t')$ and $G(Fi \rightarrow \phi) \in \Sigma$, this means that $\mathbf{M}^f \models Fi \rightarrow i[E(t')]$. But $E(t') <_f E(t)$, and as $\mathbf{M}^f \models i[E(t)]$ and $Fi \in \Sigma$, we have that $\mathbf{M}^f \models Fi[E(t')]$, which by modus ponens yields $\mathbf{M} \models i[E(t')]$. But this means that $E(t') = E(t)$, and thus all points in the filtration denoted by nominals are simple clusters. $\Box$

Again via a 'search through finite structures argument' this result is the key to the decidability of PO. It also enables us to prove the decidability for various extension; here I'll merely note that as filtrations respect trichotomy the method gives us a result for total orders as well. Thus:

Corollary 5.3 PO and LIN are decidable. $\Box$

It's worth remarking that the simple nature of the above arguments gives us another reason for being interested in COV elimination results. The proofs hinge on shifting attention from frames to models: but COV does not preserve validity in models. (For example from $\mathbf{M} \models i \rightarrow p$ we cannot conclude $\mathbf{M} \models p$.) Thus the soundness result does not go through for the finite models built by the above method, and we run the risk of falsifying theorems on them. Thus some extra work would be required to adapt the above method for systems containing COV. Finally, the above method is also useful when NTL is extended with a primitive universal modality $\Box$. Full details of such results are given in [3].

6 Concluding remarks

The basic system of NTL discussed in this paper is an interesting tool for the logical analysis of temporal expressions in natural language because it overcomes one of the greatest shortcomings of Priorean tense logic for this application — its inability to refer to times — while retaining its simplicity.

Natural language tenses are normally referential. For example, an utterance of the (simple past tense) sentence "Bill coughed" means that at some contextually determined past time *Bill cough* is true; while an utterance of the (past perfect tense) sentence "Sally had sneezed" picks out some past time and insists that before that time *Sally sneeze* is true. The referential force of these sentences can be modelled using nominals: we can represent the first sentence by $P(i \wedge \mathit{John\ cough})$ and the second by $P(i \wedge P(\mathit{Sally\ sneeze}))$. This idea extends to the analysis of simple texts. For example we might analyse "It was raining. The sky was completely grey" as follows:

$$\begin{array}{ll} & P(i \wedge \mathit{It\ be\ raining}) \\ \wedge & P(i \wedge \mathit{The\ sky\ be\ completely\ grey}). \end{array}$$

The way nominals are used here to build up representations of intersentential anaphora is reminiscent of the use of reference markers in temporal DRT [19]. More generally, NTL permits Reichenbachian insights to be incorporated into tense logic; see [4] for an exploration of these ideas.

Some relatively simple extensions enable this account to be improved; two such extensions are *relativisation to context* and *interval based semantics*. By adjoining a set of primitive contexts to frames, each assigned a time, one can introduce special atomic symbols *now*, *yesterday*, *today* and *tomorrow* which mimic rather well the locutions 'now', 'yesterday', 'today', and 'tomorrow'. The underlying semantic ideas are those of [15] and [16], but the contextualised semantics is no longer exploited using additional operators but by these new propositional variables subject to interpretational constraints. These systems provide a clean model of the basic facts about temporal reference and its interaction with tense; in particular, the scoping problems that tend to occur in multiple operator accounts are avoided. Secondly, one can move to a richer interval based semantics. Again one introduces nominals, though in the new setting nominals name out a unique interval of time, not necessarily a point. This permits finer grained analyses of temporal expressions to be given. Once again the reader is referred to [4] for a discussion of these issues.

But for present purposes the details of these extensions is not particularly important. Rather, what should be noted is the recurrence of the basic device used in this paper: the use different sorts of atom in our object languages whose interpretation is constrained in some fashion. Each sort of atom is the bearer of a different sort of referential information — we can 'read off' information just by knowing its sort — yet all this information is combined in a regular fashion by our usual connectives and operators. The idea of constraining the interpretation of variables in intensional languages is not new; it's the idea underlying general frames for example. What makes the idea interesting here is that these distinctions are syntactically marked in our object languages. As we have seen with nominals, this gives rise to sublanguages with differing logical properties; the task of charting the behaviour of such systems seems worthwhile.

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