The Logic of Correct Description

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Abstract

Austin's theory of truth is formulated in terms of a relation of *correct description* holding between a sentence and a situation. A recursive definition of correct description is provided for first-order languages containing terms denoting situations and a predicate denoting correct-description. We examine a very strong logic of situations, by restricting our attention to *situated consequence* between descriptions of *omniscient situations*, arguing that weaker logics may be obtained using standard methods. Rules of natural deduction for the logic are introduced by way of examples of natural reasoning using spatial indexicals. Finally, a Gentzen-style sequent calculus is offered.

Keywords: Austin, truth, correct description, situation, indexical, natural deduction, sequent calculus.

1 Truth and Correct Description

A basic tenet of situation semantics ([BP83, BE87, Bar89]), deriving from Austin's theory of truth ([Aus50]), is that every statement is about a situation. To make a statement by uttering the sentence 'Michel tossed the salad,' I must also be referring to a specific situation—in this case, an event—which, if my statement is true, is one in which Michel tossed the salad.

There are problems with Austin's account of truth. To say exactly how a particular situation is picked out as the reference of a given statement is notoriously hard. Austin suggested that this is achieved by what he called the *demonstrative conventions* of language-use, but he said little about what such conventions are, or how they succeed in identifying the right situation. The theory is most convincing for true statements about concrete events, such as the salad-tossing mentioned above. It fairs less well when applied to false statements ('Albert tossed the salad'), negative statements ('Jon doesn't own an electric toothbrush'), and has serious problems with universal statements ('Dictionaries are heavier than poetry books') and non-empirical statements ('$2 + 2 = 4$').

For these and similar reasons, Austin's theory has been heavily criticised, and a response to the critics is clearly needed. However, the purpose of this paper is not to give a defense of Austin. Instead, we wish to examine the consequences of the theory for logic, asking what, if anything, would change in our analysis of logical consequence if Austin's theory were true.

One obvious difficulty arises from the fact that statements, not sentences, are regarded as the primary bearers of truth. For logic, this would matter little if the truth-conditions of a statement depended only on which sentence was used in making the statement, but on Austin's account, at least in principle, the situation to which the statement refers must also be considered.

For this reason, we supplement Austin's theory, by introducing the more overtly relational semantic predicate of *correct description*:

> A sentence φ is a correct description of a situation s iff any statement about s using sentence φ would be true.

For example, the sentence 'Michel tossed a salad' correctly describes those salad-tossing events of which Michel is the agent, because any statement made about one of those situations using the sentence would be true.

Turning this definition on its head, we will take correct description to be the primary semantic relation, and define the truth of a statement as follows:

> A statement about s using sentence φ is true iff φ is a correct description of s.

For example, a statement about a specific event s made by uttering 'Michel tossed a salad' is true just in case s is one of those events which the sentence correctly describes.

It might be objected that in focusing on correct description we have departed from Austin's account in an important respect. Austin maintains that a statement is related to a type of situation by *descriptive conventions* of language-use, and that the statement is true if the situation to which it refers is of the specified type. For our purposes, this formulation has the disadvantage that one has to refer explicitly both to statements and to types of situations. By rephrasing the account in terms of correct description, one can study the relational character of the semantic theory without becoming entangled in questions about the ontology of statements and situation-types.

A potential disadvantage of this move is that the two accounts are equivalent only on the assumption that the type of situation specified by the descriptive conventions depends only on the sentence used in making the statement. In that case, we may suppose that the type of situation specified by the descriptive conventions is the type of situation correctly described by the sentence used in making the statement. However, the assumption fails for sentences containing indexicals, demonstratives, or any other elements whose meaning depends on extra-linguistic factors of the context in which they are used.

Once this problem is recognized, it poses no great threat. Our concern is to study the logic of a theoretical language for reasoning about situations, based on semantic principles deriving from Austin's analysis. But that language need not have all the features of natural language. Like Frege, we may ban the use of indexicals and other context-dependent expressions, at least for scientific purposes. This is not to say that we abandon the hope of providing an analysis of such expressions *within* the theory. One of the main purposes of a language for reasoning about situations is the development of a theory of meaning (Situation Semantics, [BP83]), but that is secondary project.

2 A Definition of Correct Description

In the previous section we defined truth in terms of the relation of correct description between sentences and situations. In this section, we will provide a recursive definition of correct description, and so complete the definition of truth. The definition is in the manner of Tarski's definition of truth for first-order languages, but is motivated by Austin's analysis instead than Tarski's. For languages without semantic vocabulary, this involves only minor changes. They become significant when the language is extended to enable one to talk about the relation of correct description itself.

We start with an ordinary first-order language, containing function-symbols and relation-symbols of arbitrary arity, with a distinguished binary relation-symbol for identity, and a countably infinite number of variables. We suppose that each situation is associated with a determinate set of objects consisting of those objects occurring in it. One object may occur in many different situations, and have different properties in each; thus the reference of predicates may vary from situation to situation. The reference of terms is invariant from one situation to another: each constant-symbol names a unique object, which may occur in different situations, and each function-symbol refers to a unique function mapping objects to other objects.

We are already in a position to define correct description for atomic sentences:

> s is correctly described by $\ulcorner R(t_1, \ldots t_n)\urcorner$ iff the objects $\underline{t_1}, \ldots \underline{t_n}$ stand in the relation $\underline{R}$ in s.[1]

But this is not enough. We also need a way of dealing with formulae containing free variables. Given a sequence σ of objects and a term t, let $t[\sigma]$ be the object which t would denote if each variable x_i occurring in t were taken to be a name for the ith object in σ. More precisely, $x_i[\sigma]$ is the ith object in the sequence σ, and $\ulcorner f(t_1, \ldots, t_n)\urcorner[\sigma]$ is a the object to which the function $\underline{f}$ maps the objects $t_1[\sigma], \ldots, t_n[\sigma]$. Now we may define correct description of atomic formulae:

> $\ulcorner R(t_1, \ldots t_n)\urcorner$ is a correct description of σ in s iff the objects $t_1[\sigma], \ldots t_n[\sigma]$ stand in the relation $\underline{R}$ in s.

A fundamental problem concerns the interpretation of negation. It is unclear how the conditions for a sentence $\ulcorner\neg\varphi\urcorner$ to describe a situation correctly depend on the conditions for φ. About all one can say in general is that it is inconsistent to say that both φ and $\ulcorner\neg\varphi\urcorner$ correctly describe the same situation. Beyond that different stories may be told to justify a range of logical principles. The classical principle,

> $\ulcorner\neg\varphi\urcorner$ correctly describes s iff φ does not correctly describe s

is the strongest, and may be thought too strong. Consider the event s of Michel tossing a salad, while preparing a light lunch in his kitchen. If he performs this culinary feat unassisted, then it fair to say that the sentence 'Albert chopped mushrooms' does not correctly describe s. But what are we to make of the sentence 'Albert did not chop mushrooms'? On the classical view, this sentence describes s correctly, even if Albert was busy chopping mushrooms in another kitchen at exactly the same time.

It is worth noting that the classical view is not entirely unwelcome here. There *is* a sense in which the situation is correctly described by the sentence 'Albert did not chop mushrooms,' if the situation is taken to be the ultimate arbitrator of mushroom-chopping in the vicinity. This observation suggests that our language might usefully be strengthened with a classical negation. Alternatively, it might be part of the meaning of particular predicates that they obey the classical principles, even if classical reasoning is inappropriate in the general case. We will see an example of this in a little while.

We opt for a minimal solution, supposing that for each n-ary predicate R, there is another n-ary predicate $\sim R$, which one may read as

[1] We shall follow the convention of using corner-quotes in the text, but ignoring them when formulae are displayed in proofs. The reference of an expression is indicated using underlining.

'not-R.' We will not legislate on the matter of which objects lie in the relation denoted by $\sim R$ in a given situation, except in so far as to claim that it is inconceivable for a sequence of objects to stand in both the relation $\underline{R}$ and $\underline{\sim R}$.

We may now give a recursive definition of what in general it is for a formula φ to be a correct description of a sequence σ of objects in a situation s. Then, we say that a sentence φ *correctly describes* s iff φ is a correct description of every sequence σ of objects in s.

For each s and sequence σ of objects, we have the following:

1. For each n-ary predicate R, and terms $t_1, \ldots, t_n$, $\ulcorner R(t_1, \ldots, t_n)\urcorner$ is a *correct description* of σ in s iff the objects $t_1[\sigma], \ldots, t_n[\sigma]$ stand in the relation denoted by R in s.
2. For each n-ary predicate R, and terms $t_1, \ldots, t_n$, $\ulcorner \neg R(t_1, \ldots, t_n)\urcorner$ is a *correct description* of σ in s iff the objects $t_1[\sigma], \ldots, t_n[\sigma]$ stand in the relation denoted by $\sim R$ in s.
3. $\ulcorner \varphi \wedge \psi \urcorner$ is a *correct description* of σ in s iff both φ and ψ are correct descriptions of σ in s.
4. $\ulcorner \neg(\varphi \wedge \psi)\urcorner$ is a *correct description* of σ in s iff either $\ulcorner \neg\varphi \urcorner$ or $\ulcorner \neg\psi \urcorner$ is a correct description of σ in s.
5. $\ulcorner \varphi \vee \psi \urcorner$ is a *correct description* of σ in s iff either φ or ψ is a correct description of σ in s.
6. $\ulcorner \neg(\varphi \vee \psi)\urcorner$ is a *correct description* of σ in s iff both $\ulcorner \neg\varphi \urcorner$ and $\ulcorner \neg\psi \urcorner$ are correct descriptions of σ in s.
7. $\ulcorner \forall x_i \varphi \urcorner$ is a *correct description* of σ in s iff φ is a correct description in s of each sequence σ' of objects differing from σ in at most the ith place.
8. $\ulcorner \neg\forall x_i \varphi \urcorner$ is a *correct description* of σ in s iff $\ulcorner \neg\varphi \urcorner$ is a correct description in s of some sequence σ' of objects differing from σ in at most the ith place.
9. $\ulcorner \exists x_i \varphi \urcorner$ is a *correct description* of σ in s iff φ is a correct description in s of some sequence σ' of objects differing from σ in at most the ith place.
10. $\ulcorner \neg\exists x_i \varphi \urcorner$ is a *correct description* of σ in s iff $\ulcorner \neg\varphi \urcorner$ is a correct description in s of each sequence σ' of objects differing from σ in at most the ith place.

There is not much to surprise and excite in this definition. In fact, for each situation s, if φ is a formula all of whose closed terms refer to objects in s, then a systematic replacement of the phrase 'is a *correct description* of σ in s' by 'is satisfied by σ' would yield the definition of satisfaction for partial first-order logic, as suggested by Kleene [Kle52].

To the basic language, we add a binary operator δ which takes a sentence as its first argument and a term as its second. If φ is a formula and t is a term, then for each s and each sequence σ,

11. $\ulcorner\delta(\varphi, t)\urcorner$ is a *correct description* of σ in s iff $t[\sigma]$ occurs in s and φ is a correct description of σ in $t[\sigma]$
12. $\ulcorner\neg\delta(\varphi, t)\urcorner$ is a *correct description* of σ in s iff $t[\sigma]$ occurs in s and φ is not a correct description of σ in $t[\sigma]$.

In particular, if φ is a sentence and t occurs in s then $\ulcorner\delta(\varphi, t)\urcorner$ is a correct description of s iff φ is a correct description of $\underline{t}$.

Implicit in the above is the assumption that the terms of the language range over situations as well as objects. We have not distinguished between situation-terms and object-terms in the syntax of the language, and so the definition of correct description applies to ordinary objects as well as to situations. Whether or not such descriptions make sense depends on the basic vocabulary of the language. One way of incorporating object-descriptions in a sensible way would be to allow property names to occur as zero-arity predicates. With 'red' and 'blue' in the language, the expression 'red$\vee$ blue' describes an object a correctly just in case a is either red or blue.

Nonetheless, there is a real distinction between situations and other objects; it lies in the fact that only situations may have other objects occurring within them. We say that

> s is a *situation* iff there is some object which occurs in s.

The definition is purely stipulative, but useful. It enables us to distinguish between situations and non-situations on the basis of the sentences describing them. A predicate of arity zero just expresses a property of the things it describes, and so may apply to situations and non-situations alike. But if φ is a sentence containing a predicate-symbol of positive (i.e., non-zero) arity, and φ correctly describes s then s is a situation. For example, if $\ulcorner R(t_1, t_2)\urcorner$ is a correct description of s then $\underline{t_1}$ and $\underline{t_2}$ occur in s, and so s is a situation.

A consequence of the description-conditions for δ-sentences is that if $\ulcorner\delta(\varphi, t)\urcorner$ correctly describes a situation s, then $\underline{t}$ occurs in s. Situations which contain other situations are called *semantic* situations. If s is a semantic situation, then for each term t referring to a situation $\underline{t}$ occurring in s and each formula φ,

> $\ulcorner\neg\delta(\varphi, t)\urcorner$ is a correct description of s iff $\ulcorner\delta(\varphi, t)\urcorner$ is not a correct description of s

In other words, matters of semantic fact are treated classically.

Finally, we will need some way of naming the current situation or object—the thing that the statement is about. We do this by allowing terms to occur as formulae. A term t may be used as a formula to say that the reference-situation (the situation to which the whole statement refers) is $\underline{t}$.

13. t is a *correct description* of σ in s iff $t[\sigma]$ is identical to s
14. $\ulcorner \neg t \urcorner$ is a *correct description* of σ in s iff $t[\sigma]$ and s are distinct.

Again, we assume that our language behaves classically with respect to terms occurring as formulae. The description-conditions for a complex formula containing terms as subformulae is given by the other clauses of the definition. For example, the formula $\ulcorner t \wedge \varphi \urcorner$ is a correct description of σ in s iff $s = t[\sigma]$ and φ is a correct description of σ in s.[2]

3 Situated Consequence and Indexicality

In assessing the validity of arguments, we need a semantic analysis of logical relationships among sentences. In particular, we are interested in characterizing when a move from sentence φ to sentence ψ is *truth-preserving*. But if truth is a property of statements, rather than sentences, how are we to determine if this is the case? For all that has been said, φ and ψ may be used to make statements about quite different situations.

To see the problem more clearly, consider the sentences 'Michel tossed a salad gracefully' and 'Michel tossed a salad.' To justify the inference from the first of these to the second, truth-preservation alone is not enough, because the two sentences may be used to make statements about different occasions, one in which Michel did toss a salad gracefully, and the other in which he did not toss a salad at all.

The problem is easily solved by basing consequence on correct description instead of truth. We say that one sentence is a *(situated) consequence* of another if every conceivable situation correctly described by the one is also correctly described by the other. In our example, 'Michel tossed a salad' is a situated consequence of 'Michel tossed a salad gracefully,' because every conceivable situation in which Michel tossed a salad gracefully, is one in which he tossed a salad.

We use the expression 'situated consequence' rather than 'logical consequence' to preserve the traditional association between the latter and the property of truth, which—on the present account—can only be had by statements. This may seem somewhat pedantic. After all, it is common to use the latter expression for consequence between

[2]The use of terms as formulae stems from Prior [Pri67] and has been more recently investigated by Blackburn [Bla90].

sentences containing indexical items or demonstratives, which rarely have a truth-value independent of their context of use.

Despite a superficial similarity between the two cases, they are really quite different. Sentences containing context-dependent elements, such as indexicals and demonstratives, are regarded as incomplete only in so far as they lack a reference for those elements. The inadequacy is a local matter, to be settled by specific aspects of the context in which they are used: 'I' and 'you' refer to the speaker and addressee; 'here' refers to the location of the speaker. Moreover, according to the traditional view, sentences without such elements do not depend on the context to provide them with a truth-value.

By contrast, according to Austin, *every* statement refers to a situation, and this reference is required to determine truth-value. The result is a much more radical departure from previous theories of truth. To appreciate the subtleties it is worth considering the statement

(S) The sentence 'Michel tossed a salad' correctly describes the situation s.

On the Austinian theory, (S) can only be true if it refers to a situation s' which is correctly described by the sentence "The sentence 'Michel tossed a salad' correctly describes the situation s." It is difficult to say exactly which situation s' is, but it must be something which involves facts about language and meaning, and so is unlikely to be identical to s, a mere salad-tossing event. Whether or not this results in a problematic regress, a hierarchy of semantic concepts, or genuine semantic relativism is a matter for further consideration, but it is safe to say that nothing like this occurs in the analysis of indexicals.

This is not to say that the situation to which a statement refers is never determined indexically. An example, which will be used extensively in Section 5, is the indexicality of many expressions with regard to spatial location. Statements made using the sentence 'There is a chill in the air' and 'It's night-time' usually refer to the situation in the vicinity of the speaker, and may often be paraphrased by 'there is a chill in the air here' and 'it's night-time here.' In such cases, the reference of the statements may be said to be determined indexically.

Yet even in these cases, one can see a distinction between the indexically determined location and the reference-situation. If, for example, we are watching a TV news-broadcast of a reporter shivering outside the state capitol, then we may make statements using the sentences 'There is a chill in the air' and 'It's night-time,' which are about the reporter's situation, rather than ours, and which may not be paraphrased using 'here.'

The distinction between the reference of indexicals in statements and the reference of the statements themselves is most clearly illustrated by the use of the past tense. Reichenbach famously distinguished between speech-time and reference-time, and used the distinction to provide a semantic classification of the tenses. In the terms of the present account, the reference-time of a statement is just the time of the event to which the statement refers, and so is determined by "demonstrative conventions" which may or may not have an indexical element.[3] By contrast, the determination of speech-time is purely indexical; it is just the time at which the statement is made.

We may also use past tense statements to bring out the difference between situated consequence and logical consequence. The statement that

(1) Albert started making the mayonnaise

about an event e is a logical consequence of the statement that

(2) Albert had almost finished making the mayonnaise.

which is about a different event e' in the later stages of the culinary process initiated by e. The truth of (1) is necessitated by the truth of (2) because there can be no event which is correctly described by the sentence used in (2) without there being another, appropriately related, event which is correctly described by the sentence used in (1). However, e and e' cannot be the same event: even a cook of Albert's great virtuosity cannot start and have almost finish making mayonnaise at once.[4] For this reason, the sentence used in (1) is *not* a situated consequence of that used in (2).

To evaluate whether a sentence ψ is a situated consequence of a sentence φ, one must ask whether there is conceivable situation s which is correctly described by φ but not by ψ; if there is then it isn't, and if there isn't then it is. It is a commonplace that such tasks of imagination are best analyzed by providing a mathematical model of (conceivable) situations and providing a rigorous definition of a relation $\models$ between models of situations and sentences, such that if m models a conceivable situation s then $m \models \varphi$ iff φ is a correct description of s. We hope that it is clear to the reader how this could be done for the language introduced in the previous section. We will not fill in the details, in part because of they are routine, but also because the level of generality involved will not be needed. In the next section we will impose some severe restrictions on the domain of "conceivable situations."

[3]In the case of past-tense statements, there is always *some* indexical element, because the event described is required to have occurred before the statement is uttered; but such constraints are very far from sufficient for reference.

[4]The analysis of the meaning of these statements is by no means uncontroversial. See [ST94] for more details.

4 Omniscient Situations

The language, or rather class of languages, introduced in Section 2 was designed to be reasonably close to what would be required for expressing a theory of situations at the elementary level—no higher-order entities are involved. It differs from previous languages for talking about situations (notably Westerstahl [Wes90]), in that we analyze the truth-conditions of statements using Austin's theory of truth, instead of Tarski's. Moreover, because of the foundational nature of the subject matter, the language is designed to reflect this semantic analysis, by including a predicate for correct description, rather than the more usual predicate of 'support' between a situation and some non-linguistic entity, like an "infon" or "state of affairs" (e.g., [Fer90],[Bar89]).

The language is very flexible and full of logical intricacies: a weak negation, restricted quantifiers, etc. Each of these is motivated by the general analysis of correct description, and the role required of such a language in expressing a theory of situations. However, a full treatment of these matters would be both overly complicated and somewhat tedious. Partial logic is now sufficiently well-understood for it to be desirable to side-step its detailed consideration if at all possible, and that is what we shall do.

Subtracting partiality, and with it restricted quantification, we are left with the δ predicate itself, terms which occur as formulae, and classical predicate logic with identity. All predicates will be assumed to behave classically: for each n-ary predicate R, objects $a_1, \ldots, a_n$, and situation s,

> $a_1, \ldots, a_n$ stand either in the relation denoted by R in s or
> in the relation denoted by $\sim R$ in s.

We may characterize the results of this surgery in semantic terms by restricting our attention to *omniscient situations*: situations in which all other situations occur. Only terms denoting omniscient situations will be used, the quantifiers will be taken to range over omniscient situations, and for a situation to count as "conceivable," it will have to be omniscient. Working with this restriction has two effects. Every object is required to be a situation, and every situation can "see" every other situation. Under these assumptions, if a situation s is correctly described by φ, then every other situation can see that this is so, and is correctly described by $\ulcorner\delta(\varphi, s)\urcorner$.

Implausible as these restrictions are in the general case, they allow us to study the logic of correct description in an environment free from the distractions of partiality. What's more, the logic we will be considering is considerably stronger that its more realistic cousins, and so suggestions of inconsistency rebuffed here have no chance of affecting the weaker logics. Finally, it is not difficult to reintroduce partiality; standard techniques are available in both model-theory and proof-theory ([Lan88, Lan89]).

In subsequent sections of the paper, we will look more closely at the proof-theory of the restricted language, but for now it it useful to give a straightforward axiomatization. First, we note that all principles of reasoning from classical first-order logic are sound in this setting. More precisely, temporarily regarding the new formulae—those of the form $\ulcorner\delta(\varphi, t)\urcorner$ and t—as atomic formulae, we may say that

> if a formula φ is a classical consequence of a set of formulae Γ, then φ is a situated consequence of Γ.

On top of this classical base, we may add all instances of the following schema:

1. $\forall x(\delta(\varphi \wedge \psi, x) \leftrightarrow \delta(\varphi, x) \wedge \delta(\psi, x))$
2. $\forall x(\delta(\varphi \vee \psi, x) \leftrightarrow \delta(\varphi, x) \vee \delta(\psi, x))$
3. $\forall x(\delta(\neg\varphi, x) \leftrightarrow \neg\delta(\varphi, x))$
4. $\forall x(\delta(\forall y\varphi, x) \leftrightarrow \forall y\delta(\varphi, x))$
5. $\forall x(\delta(\exists y\varphi, x) \leftrightarrow \exists y\delta(\varphi, x))$
6. $\forall x\forall y(\delta(x, y) \leftrightarrow x = y)$
7. $\forall x\forall y(\delta(\delta(\varphi, y), x) \leftrightarrow \delta(\varphi, y))$

(with telescoping abbreviations: $\ulcorner\varphi_1 \rightarrow \varphi_2\urcorner$ for $\ulcorner\neg\varphi_1 \vee \varphi_2\urcorner$ and $\ulcorner\varphi_1 \leftrightarrow \varphi_2\urcorner$ for $\ulcorner\varphi_1 \rightarrow \varphi_2 \wedge \varphi_2 \rightarrow \varphi_1\urcorner$).

That instances of the first seven axiom schema correctly describe every omniscient situation follows directly from the definition of correct description given in Section 2. For the last schema, we also need the requirement that every situation is omniscient.

The completeness of the above axiomatization may be demonstrated by a straightforward modification of the proof of the completeness of classical first-order logic. Let Σ be a sets of sentences which are classically consistent w.r.t. the above axioms, and let Σ^* be a Henkin-complete and consistent extension of Σ. We know that such sets exist by the usual argument from Zorn's Lemma. Now for each closed term t, "construct" a situation t^* such that

1. $t_1^* = t_2^*$ iff the sentence $\ulcorner t_1 = t_2\urcorner$ is in Σ^*, and
2. objects $t_1^*, \ldots, t_n^*$ stand in the relation denoted by R in t^* iff the sentence $\ulcorner\delta(R(t_1, \ldots, t_n), t)\urcorner$ is in Σ^*.

If we had given a formal semantics for our language, we could perform this construction using the usual set-theoretic methods, given the closure of Σ^* under the usual laws of identity. In any case, we only require that for each term t, there is a conceivable situation satisfying the above; and that seems unproblematic. It only remains to show that for each sentence φ and each t,

$$\varphi \text{ is a correct description of } t^* \text{ iff } \ulcorner\delta(\varphi, t)\urcorner \text{ is in } \Sigma^*.$$

The proof of this fact follows the similar proof for classical first-order logic, by induction on the logical complexity of φ, using each of the axioms schematized above. The axioms directly reflect the conditions for correct-description, given the completeness of classical first-order logic, and so it is unsurprising that they are also complete.

As an illustration of how the restricted languages may be of use, despite their limitations, consider a language containing the binary predicate '$\trianglelefteq$,' used to denote the relation of one situation's being part of another. The requirement that $\trianglelefteq$ behave classically does not seem to be too repugnant.

The theory of $\trianglelefteq$ may be developed in various ways, but a central concern is that of the persistence of information from a situation to one containing it. We may express the fact that a formula φ is persistent by

$$\ulcorner\forall x \forall y\ (\delta(\varphi, x) \wedge x \trianglelefteq y) \rightarrow \delta(\varphi, y)\urcorner$$

According to one school of thought, every sentence is persistent. Call this the Strong Theory of persistence. It is axiomatized by all sentences instantiating the above schema.

It should be clear that the Strong Theory is rather too strong in the present setting. For suppose that $s_1 \trianglelefteq s_2$ and that s_1 and s_2 are distinguishable. Then there is a sentence φ which correctly describes s_2 but not s_1, and so $\ulcorner\neg\varphi\urcorner$ describes s_1 correctly but not s_2, contradicting the claimed persistence of $\ulcorner\neg\varphi\urcorner$. Thus the Strong Theory entails that all $\trianglelefteq$-related situations are indistinguishable.

There are independent reasons for rejecting the Strong Theory ([Bar89],[Coo91],[CK91]), so we need not be too sad about it's failure in the present context. Moreover, something of persistence may be salvaged by restricting it to formulae not containing negation. In fact, we may do a little better. We have already seen how to state the persistence of a given formula; now we define the *anti-persistence* of φ by

$$\ulcorner\forall x \forall y\ (\delta(\varphi, x) \wedge y \trianglelefteq x) \rightarrow \delta(\varphi, y)\urcorner$$

Now observe that

1. $\ulcorner\neg\varphi\urcorner$ is persistent iff φ is anti-persistent
2. $\ulcorner\neg\varphi\urcorner$ is anti-persistent iff φ is persistent

3. $\ulcorner\varphi \wedge \psi\urcorner$ and $\ulcorner\varphi \vee \psi\urcorner$ are persistent if φ and ψ are persistent
4. $\ulcorner\varphi \wedge \psi\urcorner$ and $\ulcorner\varphi \vee \psi\urcorner$ are anti-persistent if φ and ψ are anti-persistent
5. $\ulcorner\exists x \varphi\urcorner$ and $\ulcorner\forall x \varphi\urcorner$ are persistent if φ is persistent
6. $\ulcorner\exists x \varphi\urcorner$ and $\ulcorner\forall x \varphi\urcorner$ are anti-persistent if φ is anti-persistent

We may dub this the Conditional Theory of persistence. It shows how any initial classification of atomic formulae according to their persistence properties may be extended to a wide class of other formulae.

5 Situated Reasoning: The Spatial Analogy

In Section 3 we mentioned that the situation to which a statement refers is sometimes fixed by an indexically determined spatial location. Paradigm examples are statements concerning the weather and other prevailing conditions. In this section we will focus exclusively on such statements, using them to motivate natural rules of deduction.

So, from this point on, until the end of the section, we will assume that the situation described by a statement is the one the stater is *in*. Thus, the statement that it's raining refers to a suitably encompassing situation, determined by my present location, which happens to be Bloomington, Indiana. The situation is correctly described by the sentence 'It's raining' just in case it's raining in Bloomington, Indiana.

This last equivalence permits us to look to our usual patterns of reasoning using sentence of the form $\ulcorner$In l, $\varphi\urcorner$ or $\ulcorner\varphi$ in $l\urcorner$ (where l is a term referring to a spatial location) in order to motivate more general principles of reasoning about correct description. Furthermore, sentences of the form $\ulcorner$This is $l\urcorner$ play an analogous role to our situation-terms: they correctly describe a situation iff it is the situation determined by the location l. These two observations form the basis of what we shall call the Spatial Analogy.

If t_l is a term referring to the situation determined by the location l, we capture the central correspondence of the Spatial Analogy as follows:

1. $\ulcorner$In l, $\varphi\urcorner$ is equivalent to $\ulcorner\delta(\varphi, t_l)\urcorner$,
2. $\ulcorner$This is $l\urcorner$ is equivalent to t_l, and
3. $\ulcorner l_1$ is $l_2\urcorner$ is equivalent to $\ulcorner\delta(t_{l_1}, t_{l_2})\urcorner$.

Our method will be to examine how the expressions on the left hand side are used in constructing valid arguments, and to formulate natural-deduction rules which characterize those arguments. By analogy, the rules will transfer to rules involving the expressions on the right hand side. Particular attention will be given to arguments which either "introduce" a spatial expression in the conclusion which does not occur in the premises or "eliminate" an expression which does occur in the premises by drawing a conclusion which does not contain the expression. For the primary spatial word 'in' the following arguments are examples of introduction and elimination:

(1) The sun is shining; this is Bloomington, so the sun is shining in Bloomington.

(2) In Tokyo, people drive on the left; this is Tokyo, so people drive on the left.

They suggest the following introduction and elimination rules for 'in':

$$\dfrac{\varphi \qquad \text{This is } l}{\text{In } l,\ \varphi}\,\mathit{In}\text{-I} \qquad\qquad \dfrac{\text{In } l,\ \varphi \qquad \text{This is } l}{\varphi}\,\mathit{In}\text{-E}$$

These rules should be supplemented with a natural-deduction calculus for classical logic (such as that found in [?]) so that the logical connectives behave in the usual way. Are they sufficient? Unfortunately not. The rules for 'in' are sound—no invalid arguments can be made using them—but there are intuitively valid arguments which cannot be represented. The most important arguments are those that depend on the reasoner imagining, as a hypothetical premise, that they are somewhere else. For example:

(3) Alcohol is forbidden in Abu Dabi; Sake contains alcohol; so Sake is forbidden in Abu Dabi.

This argument can be justified using the introduction and elimination rules for 'in' if, in addition, the hypothetical premise 'This is Abu Dabi' is allowed for the duration of the argument:

$$\dfrac{[\text{This is Abu Dabi}] \qquad \dfrac{\dfrac{\text{In Abu Dabi, alcohol is forbidden} \qquad [\text{This is Abu Dabi}]}{\text{Alcohol is forbidden}}\,\mathit{In}\text{-E} \qquad \text{Sake contains alcohol}}{\text{Sake is forbidden}}\,\text{C.S.}}{\text{In Abu Dabi, sake is forbidden}}\,\mathit{In}\text{-I}$$

(The rule C.S. is an abbreviation for a sub-proof which could be constructed from a "common sense" understanding of the logical form of the example sentences.) At the last step of the proof, the hypothetical premise is *discharged*—removed from the set of genuine premises—and this is indicated by enclosing any occurrences of the premise in brackets.

The strategy works well for arguments like (3), but it may lead us to make mistakes if applied more widely. For example, the invalid argument

(4) In Islamabad, it only rains during monsoon; it's raining; so it's monsoon in Islamabad

appears to have a similar proof:

$$\dfrac{[\text{This is Islamabad}] \qquad \dfrac{\dfrac{\text{In Islamabad, it only rains during monsoon} \qquad [\text{This is Islamabad}]}{\text{It only rains during monsoon}}\,\mathit{In}\text{-E} \qquad \text{It's raining}}{\text{It's monsoon}}\,\text{C.S.}}{\text{In Islamabad, it's monsoon}}\,\mathit{In}\text{-I}$$

It is easy to see what is going wrong. The premise 'It's raining' is spatially-indexical: said here, one may infer that it's raining here but not that it's raining in Islamabad. On hypothetical journeys one should avoid burdening oneself with premises which are true at home but false abroad.

This kind of mistake may be avoided if care is taken in specifying the circumstances under which a hypothetical premise of the form $\ulcorner$This is $\bar{l}\urcorner$ may be discharged. A sufficient condition is that the "context" of the proof—the set of premises and conclusion after the discharge—does not contain any spatially-indexical sentences. This condition is met in the proof of argument (3) since the context consists of only 'Alcohol is forbidden in Abu Dabi,' 'Sake contains alcohol' and 'Sake is forbidden in Abu Dabi,' none of which are spatially-indexical. The condition is *not* met by the proof of argument (4) since that contains the spatially-indexical sentence 'It's raining.'

We are not done yet. There is another way of using hypothetical premises to obtain valid arguments which cannot be proved, even using the discharge rule described above. Consider the argument:

(5) It's raining; wherever it's raining, it's wet, so it's wet.

Suppose that 'wherever' is a universal quantifier ranging over locations. Then the second premise is equivalent to the following sentence:

$\ulcorner$For all locations l, if it's raining in l then it's wet in $\bar{l}\urcorner$.

In order to use this premise, a reasoner must find a location at which to instantiate it. Of course, the reasoner would like to instantiate it at her current location, where it's raining. To do so, she must first give a name to her current location—'X,' say. This may be done by assuming hypothetically 'This is X.' She may then infer 'It's raining in X' (by 'in'-introduction), instantiate the second premise to 'If it's raining in

X then it's wet in X,' deduce 'It's wet in X' and conclude 'It's wet' (by 'in'-elimination). Finally, she should discharge the hypothetical premise 'This is X':

$$\dfrac{[\text{This is } X] \qquad \dfrac{\dfrac{\text{It's raining} \quad [\text{This is } X]}{\text{In } X\text{, it's raining}}\,\textit{In}\text{-I} \qquad \dfrac{\text{Wherever it's raining, it's wet}}{\text{If it's raining in } X \text{ then it's wet in } X}\,\forall\text{-E}}{\text{In } X\text{, it's wet}}\,\to\text{-E}}{\text{It's wet}}\,\textit{In}\text{-E}$$

Such a discharge is not licensed by the condition discussed above because the context contains the spatially-indexical sentences 'It's raining' and 'It's wet.' The discharge of 'This is X' is only safe because it uses an invented name, 'X.' Considerations of this kind suggest that there is another sufficient condition on discharges of premises of the form ⌜This is l⌝: that l is a place-name which does not occur in any sentence of the context.

The discharge in the proof of (5) is licensed because its context consists of the sentences 'It's raining,' 'Whenever it's raining, it's wet' and 'It's wet,' none of which contain an occurrence of 'X.' The discharge of 'This is Islamabad' in the non-proof of (4) is *not* licensed because its context contains the sentences 'In Islamabad, it only rains in the monsoon' and 'It's monsoon in Islamabad,' both of which contain occurrences of 'Islamabad.'

The introduction and elimination rules for 'in' together with rules for the usual classical connectives and the two discharge rules mentioned above are jointly sufficient to capture all spatial consequences involving spatial expressions of the form ⌜In l, φ⌝ and ⌜This is l⌝.

The attribution of properties to places by expressions of the form ⌜$P(l)$⌝ has not been mentioned explicitly, but many of the logical properties of such expressions are captured by the usual rules of classical logic; in particular, the use of spatial quantifiers (such as 'wherever' and 'somewhere') to express generalizations concerning places can be captured by the same rules as those for individual quantifiers.

Special to the spatial case is the attribution of *relational* properties to places. Being on the other side of the world is a relation property, so expressions such as 'Tokyo is on the other side of the world' are spatially-indexical: they relate the subject, Tokyo, to the indexically-determined current location of the stater. The logical properties of expressions using relational predicates is governed by the rules given above. For example, the following examples are proved by instances of the introduction and elimination rules for 'in':

(7) This is Bloomington; Edinburgh is far away, so in Bloomington, Edinburgh is far away.

(8) In Edinburgh, London is to the south; this is Edinburgh, so London is to the South.

(9) SY 026 is across campus; the office of the Philosophy Dept. is SY 026, so the office of the Philosophy Dept. is across campus.

(7) and (8) are both provable using a single application of the *In*-elimination rule, and (9) require the principle of indiscernibility of identicals.

This completes the analysis of natural deduction using the spatial expressions $\ulcorner$In l, $\varphi\urcorner$, $\ulcorner$This is $l\urcorner$ and $\ulcorner l_1$ is $l_2\urcorner$. We have omitted much, because we are only concerned to motivate those patterns of reasoning which may be lifted to the general case. Disanalogous aspects of the correspondence have been ignored, and there are some further possibilities for the Spatial Analogy which have not been explored, such as the analogy between $\trianglelefteq$ and spatial-part.

6 Two Calculi

The patterns of reasoning displayed in the previous section may be transferred to a more general setting, by virtue of the Spatial Analogy. First, we will need a few syntactic details.

In formulating a natural-deduction calculus it is convenient to have a stock of *formal parameters*. These are terms which do not occur in any formula of the language, but which may be used in hypothetical reasoning. We use 'a' to range over parameters. A formula (term) is *parametric* if it contains a parameter; otherwise, it is *non-parametric*. A term is *closed* if it contains no free variables; it may contain parameters. Given a formula φ, a parameter a and a term t, the formula φ^a_t is the result of replacing every occurrence of a in φ by t. A δ-*formula* is any formula of the form $\ulcorner\delta(\varphi, t)\urcorner$.

In addition to the usual introduction and elimination rules for the classical connectives (as given in [?], for example), we formulate rules for the predicate δ. They are directly analogous to those for $\ulcorner$In $l\urcorner$, φ.'

$$\frac{\varphi \quad t}{\delta(\varphi, t)}\delta\text{-I} \qquad \frac{\delta(\varphi, t) \quad t}{\varphi}\delta\text{-E}$$

In addition, we have two new ways in which premises may be discharged:

Term rule If t is a term which occurs as a premise of a proof, then all occurrences of t among the premises may be discharged at once, so long as the premises and conclusion of the resulting proof are all δ-formulae.

Parameter rule If a is a parameter which occurs as a premise of a proof, then all occurrences of a among the premises may be discharged at once, so long as the premises and conclusion of the resulting proof do not contain occurrences of a.

The Term rule permits the use of terms as hypothetical premises in a proof by stating that to prove an argument, it is legitimate to make the hypothetical assumption that the situation being described is named by a given term. The rule's side-condition is analogous to the first condition on hypothetical premises of the form $\ulcorner$This is $\bar{l}\urcorner$ discussed in the previous section. The Parameter Rule enforces a condition analogous to the second condition on hypothetical spatial premises: it allows the reasoner to "name" the situation being described with a new parameter.

Finally, we need a rule capturing the fact that $\ulcorner\delta(t_1, t_2)\urcorner$ means that t_1 and t_2 have the same reference. By the Spatial Analogy, expression of the form $\ulcorner\delta(t_{l_1}, t_{l_2})\urcorner$ correspond to identity statements of the form $\ulcorner l_1$ is $l_2\urcorner$. To capture the logic of the former expressions, we will need to import rules corresponding to the classical rules for identity. Reflexivity is already derivable:

$$\dfrac{\dfrac{[t]^1 \qquad [t]^1}{\delta(t,t)}\,\delta\text{-I}}{\delta(t,t)}\,\text{Term}^1$$

But to capture the substitutivity of co-referring terms, we need the following Substitution rules:[5]

$$\dfrac{\delta(t_1,t_2) \qquad \varphi^a_{t_1}}{\varphi^a_{t_2}}\,\text{Sub-1} \qquad\qquad \dfrac{\delta(t_2,t_1) \qquad \varphi^a_{t_1}}{\varphi^a_{t_2}}\,\text{Sub-2}$$

The resulting natural-deduction calculus, which we shall call NK_δ is sound with respect to the restricted interpretation of our languages, as we hope the reader of the previous section is now convinced.[6] It is

[5]The parameter a does *not* occur in any formula occurring in an application of one of these rules. The parameter is used merely as a placeholder to indicate the position in which the terms t_1 and t_2 are placed.

[6]If any doubt remains, the rules may be checked against the definition of correct-description given in Section 2, taking into account the additional assumptions of the

complete, because all the axioms listed in Section 4 are provable. For example,

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\delta(\delta(\varphi,a),b)]^2 \quad [b]^1}{\delta(\varphi,a)}\,\delta\text{-E}}{\delta(\varphi,a)}\,\text{Term}^1}{\delta(\delta(\varphi,a),b)\rightarrow\delta(\varphi,a)}\,{\rightarrow}\text{-I}^2 \qquad \dfrac{\dfrac{\dfrac{[\delta(\varphi,a)]^4 \quad [b]^3}{\delta(\delta(\varphi,a),b)}\,\delta\text{-I}}{\delta(\delta(\varphi,a),b)}\,\text{Term}^3}{\delta(\varphi,a)\rightarrow\delta(\delta(\varphi,a),b)}\,{\rightarrow}\text{-I}^4}{\dfrac{\delta(\delta(\varphi,a),b)\leftrightarrow\delta(\varphi,a)}{\dfrac{\forall y(\delta(\delta(\varphi,a),y)\leftrightarrow\delta(\varphi,a))}{\forall x\forall y(\delta(\delta(\varphi,x),y)\leftrightarrow\delta(\varphi,x))}\,\forall\text{-I}}\,\forall\text{-I}}\,{\leftrightarrow}\text{-I}
$$

Our final offering is a Gentzen-style sequent-calculus for the same logic. Throughout, as above, 'φ' and 'ψ' range over sentences, 's' and 't' range over closed terms, 'a' ranges over parameters, and 'Γ' and 'Δ' range over sets of sentences. Standard abbreviations using ',' for set union on either side of '$\vdash$' will be used.

Structural Rules

$$
\dfrac{}{\varphi\vdash\varphi}\,\text{I} \qquad\qquad \dfrac{\Gamma\vdash\Delta}{\Gamma',\Gamma\vdash\Delta,\Delta'}\,\text{W}
$$

The familiar rules of Identity and Weakening are both sound, for the usual reasons.

Term Rules

$$
\dfrac{a,\Gamma\vdash\Delta}{\Gamma\vdash\Delta}\,\text{P}\ (a\ \text{new}) \qquad\qquad \dfrac{a,\Gamma\vdash\Delta}{\Gamma\vdash\Delta}\,\text{T}\ (\Gamma,\Delta\ \text{are}\ \delta)
$$

The P-rule is the sequent-calculus analogue of the Parameter rule. The side-condition requires a to be a parameter which does not occur in Γ or in Δ. It is sound because any situation s which is a counterexample to the validity of $\ulcorner\Gamma\vdash\Delta\urcorner$ may be named by a, so long as a is not already used for some other purpose.

The T-rule is the sequent-calculus analogue of the Term rule. The side-condition requires that Γ and Δ consist only of δ-formulae. It is sound, because any counterexample to $\ulcorner\Gamma\vdash\Delta\urcorner$ is a situation correctly described by each formula in Γ and no formula in Δ. But the formulae

restricted interpretation.

in Γ and Δ are required to be δ-formulae, so each formula in Γ and no formula in Δ is a correct description of $\underline{t}$, and so $\underline{t}$ is a counterexample to $\ulcorner t, \Gamma \vdash \Delta \urcorner$.

Substitution Rules

$$\frac{\Gamma^a_{t_1} \vdash \Delta^a_{t_1}}{\delta(t_1, t_2), \Gamma^a_{t_2} \vdash \Delta^a_{t_2}}\mathrm{S}_1 \qquad \frac{\Gamma^a_{t_1} \vdash \Delta^a_{t_1}}{\delta(t_2, t_1), \Gamma^a_{t_2} \vdash \Delta^a_{t_2}}\mathrm{S}_2$$

The S-rules are the sequent-calculus analogues of the Substitution rules. The rule S_1 is sound because if $\ulcorner \delta(t_1, t_2), \Gamma^a_{t_2} \vdash \Delta^a_{t_2} \urcorner$ has a counterexample s then $\ulcorner \delta(t_1, t_2) \urcorner$ correctly describes s and so $\underline{t_1} = \underline{t_2}$. Consequently, each formula $\varphi^a_{t_1}$ in $\Gamma^a_{t_1}, \Delta^a_{t_1}$ correctly describes s iff $\varphi^a_{t_2}$ does, and so s is a counterexample to $\ulcorner \Gamma^a_{t_1} \vdash \Delta^a_{t_1} \urcorner$

Description Rules

$$\frac{\Gamma \vdash \Delta, t \qquad \varphi, \Gamma \vdash \Delta}{\delta(\varphi, t), \Gamma \vdash \Delta}\mathrm{L}\delta\!\downarrow \qquad \frac{\Gamma \vdash \Delta, t \qquad \Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta, \delta(\varphi, t)}\mathrm{R}\delta\!\downarrow$$

$$\frac{\delta(\varphi, t), \Gamma \vdash \Delta}{t, \varphi, \Gamma \vdash \Delta}\mathrm{L}\delta\!\uparrow (\varphi \text{ is non-}\delta) \qquad \frac{\Gamma \vdash \Delta, \delta(\varphi, t)}{t, \Gamma \vdash \Delta, \varphi}\mathrm{R}\delta\!\uparrow (\varphi \text{ is non-}\delta)$$

The four rules for description are related to the natural-deduction rules δ-I and δ-E. We will demonstrate their soundness in turn.

$\mathrm{L}\delta\!\downarrow$ rule. A counterexample to a sequent of the form $\ulcorner \delta(\varphi, t), \Gamma \vdash \Delta \urcorner$ is a situation s which is correctly described by all the formulae in $\ulcorner \delta(\varphi, t) \urcorner, \Gamma$ and none of the formulae in Δ. In particular, s is correctly described by $\ulcorner \delta(\varphi, t) \urcorner$, and so $\underline{t}$ is correctly described by φ. There are two possibilities: either t names s or it doesn't. If $\underline{t} = s$ then s is a counterexample to $\ulcorner \varphi, \Gamma \vdash \Delta \urcorner$. Alternatively, if $\underline{t} \neq s$ then s is a counterexample to $\ulcorner \Gamma \vdash \Delta, t \urcorner$.

$\mathrm{L}\delta\!\uparrow$ rule. A counterexample to a sequent of the form $\ulcorner t, \varphi \Gamma \vdash \Delta \urcorner$ is a situation s which is correctly described by all the formulae in t, φ, Γ and none of the formulae in Δ. Thus $\underline{t}$ $(=s)$ is correctly described by φ, and so every situation, including s, is correctly described by $\ulcorner \delta(\varphi, t) \urcorner$. Hence s is a counterexample to $\ulcorner \delta(\varphi, t), \Gamma \vdash \Delta \urcorner$.

The right rules $\mathrm{R}\delta \downarrow$ and $\mathrm{R}\delta \uparrow$ by be seen to be sound by very similar arguments. Note that two *upward* rules, $\mathrm{L}\delta\!\uparrow$ and $\mathrm{R}\delta\!\uparrow$, fail to have the subformula property: the formula $\ulcorner \delta(\varphi, t) \urcorner$ need not occur as a subformula of any formula in the conclusion. This is unfortunate, but unavoidable (See [Sel91], for a discussion of this point). The damage is limited by the side-condition that φ be a non-δ-formula, which is not actually necessary for soundness.

Together with the standard rules for conjunction, disjunction, negation, universal and existential quantification (repeated in the Appendix) the above rules form a calculus which we call LK_δ.

That each natural deduction in NK_δ is provable in LK_δ follows from the derivability of each of the natural-deduction rules, and the following theorem. The completeness of the calculus LK_δ therefore follows from the completeness of the natural-deduction system.

Theorem 6.1 *If* $\Gamma \vdash \Delta, \varphi$ *and* $\varphi, \Gamma' \vdash \Delta'$ *then* $\Gamma, \Gamma' \vdash \Delta, \Delta'$.

A sketch of a proof of this theorem—an extension of Gentzen's *Haupstatz* for the system *LK* of classical first-order logic—is given in the Appendix.

By inspecting the rules of LK_δ, we see that the logic is almost *analytic* in the proof-theoretic sense: whether or not a sequent has a proof is determined by whether or not its parts have proofs. To make this condition more precise, we say that a formula φ is a *quasi-subformula* of a set Σ of formulae if either

1. φ is a subformula of one of the formulae in Σ, or
2. $\varphi = \ulcorner\delta(\psi, t)\urcorner$ and ψ and t are subformulae of formulae in Σ.

Theorem 6.2 *If* π *is a proof of* $\ulcorner\Gamma \vdash \Delta\urcorner$ *then there is a finite set* P *of parameters such that every formula occurring in* π *is a quasi-subformula of* $P, \Gamma, Delta$.

PROOF: Let P be the set of parameters occurring in a proof π of $\ulcorner\Gamma \vdash \Delta\urcorner$. All the rules have the subformula property except the the P-rule and the upward δ-rules. The P-rule is covered by the inclusion of P in the set P, Γ, Δ. Hence every non-δ-formula occurring in π is a subformula of some formula in P, Γ, Δ. But now consider applications of the upward δ-rules:

$$\frac{\delta(\varphi, t), \Gamma \vdash \Delta}{t, \varphi, \Gamma \vdash \Delta}\mathrm{L}\delta\uparrow \qquad\qquad \frac{\Gamma \vdash \Delta, \delta(\varphi, t)}{t, \Gamma \vdash \Delta, \varphi}\mathrm{R}\delta\uparrow$$

The formula φ is required to be a non-δ-formula, and so is a subformula of some formula in P, Γ, Δ, by our previous remark. Likewise, the term t is subformula of some formula in the same set, and so the formula $\ulcorner\delta(\varphi, t)\urcorner$ is a quasi-subformula of P, Γ, Δ. QED

The theorem may be used to prove decidability and interpolation properties of various fragments of the logic. A discussion of fragments of the logic is given in [Sel93].

7 Concluding Remarks

We have examined the concept of correct description suggested by Austin's 1950 theory of truth. In the first part of the paper we gave an analysis of the concept in the form of a recursive definition, initially for a standard first-order language, and then for a language containing terms referring to situations and a correct-description predicate 'δ'. We made some idealizing assumptions about situations and gave a complete axiomatization of the resulting logic. In the second part we studied natural patterns of reasoning using 'δ' by way of the example of spatially-indexical language. This was developed into two formal calculi: a natural-deduction calculus NK_δ and a sequent-calculus LK_δ. The latter was shown to admit Cut and to be (almost) analytic.

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Appendix: *Haupstatz* for $\mathbf{LK}_\delta$

The system $\mathbf{LK}_\delta$ (We omit the rules for implication and identity for the sake of brevity. Each may be considered to be defined in terms of the other connectives: $\ulcorner \varphi \rightarrow \psi \urcorner$ is just $\ulcorner \neg\varphi \vee \psi \urcorner$ and $\ulcorner t_1 = t_2 \urcorner$ is $\ulcorner \delta(t_1, t_2) \urcorner$. In both cases, the equivalence between definiens and definiendum is provable in the system augmented with the appropriate rules.)

$$\frac{}{\varphi \vdash \varphi}\,\mathrm{I} \qquad \frac{\Gamma \vdash \Delta}{\Gamma', \Gamma \vdash \Delta, \Delta'}\,\mathrm{W} \qquad \frac{a, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\,\mathrm{P}\ {\scriptstyle (a \text{ new})} \qquad \frac{a, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\,\mathrm{T}\ {\scriptstyle (\Gamma, \Delta \text{ are } \delta)}$$

$$\frac{\Gamma^a_{t_1} \vdash \Delta^a_{t_1}}{\delta(t_1, t_2), \Gamma^a_{t_2} \vdash \Delta^a_{t_2}}\,\mathrm{S}_1 \qquad \frac{\Gamma^a_{t_1} \vdash \Delta^a_{t_1}}{\delta(t_2, t_1), \Gamma^a_{t_2} \vdash \Delta^a_{t_2}}\,\mathrm{S}_2$$

$$\frac{\Gamma \vdash \Delta, t \qquad \varphi, \Gamma \vdash \Delta}{\delta(\varphi, t), \Gamma \vdash \Delta}\,\mathrm{L}\delta\!\downarrow \qquad \frac{\Gamma \vdash \Delta, t \qquad \Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta, \delta(\varphi, t)}\,\mathrm{R}\delta\!\downarrow$$

$$\frac{\delta(\varphi, t), \Gamma \vdash \Delta}{t, \varphi, \Gamma \vdash \Delta}\,\mathrm{L}\delta\!\uparrow\ {\scriptstyle (\varphi \text{ are non-}\delta)} \qquad \frac{\Gamma \vdash \Delta, \delta(\varphi, t)}{t, \Gamma \vdash \Delta, \varphi}\,\mathrm{R}\delta\!\uparrow\ {\scriptstyle (\varphi \text{ non-}\delta)}$$

$$\frac{\varphi, \psi, \Gamma \vdash \Delta}{\varphi \wedge \psi, \Gamma \vdash \Delta}\text{L}\wedge \qquad \frac{\varphi, \Gamma \vdash \Delta \qquad \psi, \Gamma \vdash \Delta}{\varphi \vee \psi, \Gamma \vdash \Delta}\text{L}\vee \qquad \frac{\Gamma \vdash \Delta, \varphi}{\neg\varphi, \Gamma \vdash \Delta}\text{L}\neg$$

$$\frac{\Gamma \vdash \Delta, \varphi \qquad \Gamma \vdash \Delta, \psi}{\Gamma \vdash \Delta, \varphi \wedge \psi}\text{R}\wedge \qquad \frac{\Gamma \vdash \Delta, \varphi, \psi}{\Gamma \vdash \Delta, \varphi \vee \psi}\text{R}\vee \qquad \frac{\varphi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg\varphi}\text{R}\neg$$

$$\frac{\varphi^x_t, \Gamma \vdash \Delta}{\forall x.\varphi, \Gamma \vdash \Delta}\text{L}\forall \quad \frac{\Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta, \forall x.\varphi^a_x}\text{R}\forall \text{ (a new)} \quad \frac{\varphi, \Gamma \vdash \Delta}{\exists x.\varphi^a_x, \Gamma \vdash \Delta}\text{L}\exists \text{ (a new)} \quad \frac{\Gamma \vdash \Delta, \varphi^x_t}{\Gamma \vdash \Delta, \exists x.\varphi}\text{R}\exists$$

In order to demonstrate the admissibility of Cut in the system LK_δ, we extend it to the system LK^*_δ by adding the *S-axioms*

$$\frac{}{\delta(t_1, t_2), \varphi^a_{t_2} \vdash \psi^a_{t_2}}\text{S}_1\text{ax} \qquad \frac{}{\delta(t_2, t_1), \varphi^a_{t_2} \vdash \psi^a_{t_2}}\text{S}_2\text{ax}$$

for formulae φ and ψ such that $\varphi^a_{t_1} = \psi^a_{t_1}$, and three *Cut-rules* of the form

$$\frac{\Gamma \vdash \Delta, \varphi_1 \qquad \varphi_2, \Gamma' \vdash \Delta'}{\varphi_3\Gamma, \Gamma' \vdash \Delta, \Delta'}C(\varphi_1, \varphi_2, \varphi_3)$$

given by

1. C = all instances of $\text{C}(\varphi, \varphi, \emptyset)$
2. CδL = all instances of $\text{C}(\varphi, \delta(\varphi, t), t)$
3. CδR = all instances of $\text{C}(\delta(\varphi, t), \varphi, t)$

In each case, we call φ the *cut-formula* of the rule.

We show that any proof in LK^*_δ may be replaced by a proof in LK_δ with the same end-sequent. We order formulae in complexity by the number of connectives they contain. A *cut* in a proof is an application of one of the Cut-rules. A cut c in a proof is *at least as complex as* another cut c' if

1. the cut-formula of c is at least as complex as that of c', and
2. if the cut-formula of c is of the same complexity as that of c', then the sum of the heights of the proofs of the premises c is at least as great as the corresponding sum for c'.

A proof π is *at least as complex as* another proof π' if there is a cut occurring in π which is at least as complex as all the cuts occurring in π'. A proof is *S-reduced* if it contains no application of the S-rules (although it may have S-axioms).

Claim 1 For any proof π in LK^*_δ there is an S-reduced proof π' in LK^*_δ with the same end-sequent and of no greater complexity.

PROOF OF CLAIM: It is routine to check that the S-rules commute with every other rule in LK^*_δ. Such transformations do not change the complexity of the proof. The only case worthy of suspicion is that of the T-rule, because of its side-condition; but $\delta(t_1, t_2)$ is a δ-formula, and so the rules commute without difficulty. In this way, applications of the S-rules may be pushed up the proof-tree until they reach the leaves. At the leaves, they may be replaced by S-axioms as follows:

$$\dfrac{\dfrac{}{\varphi^a_{t_1} \vdash \psi^a_{t_1}}\text{I}}{\delta(t_1, t_2), \varphi^a_{t_2} \vdash \psi^a_{t_2}}\text{S}_1 \quad \rightsquigarrow \quad \dfrac{}{\delta(t_1, t_2), \varphi^a_{t_2} \vdash \psi^a_{t_2}}\text{S}_1\text{ax}$$

where $\varphi^a_{t_1} = \psi^a_{t_1}$, as required for the I-rule and for the S-axiom. The transformation for S_2 is similar. Note that this process does not change the logical structure of the cut-formulae of any of the cuts in the proof, and so the complexity is not increased.

Claim 2 Suppose there is an S-reduced proof π in LK^*_δ ending in a cut

$$\dfrac{\begin{array}{c}\pi_1 \\ \Gamma \vdash \Delta, \varphi_1\end{array} \qquad \begin{array}{c}\pi_2 \\ \varphi_2, \Gamma' \vdash \Delta'\end{array}}{\varphi_3\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

of maximal complexity among the cuts in π. Then there is an S-reduced proof in LK^*_δ of the sequent $\varphi_3\Gamma, \Gamma' \vdash \Delta, \Delta'$ which is strictly less complex than π.

PROOF OF CLAIM: The structure of the proof is very similar to that of cut-elimination for classical logic, so we will only give a sketch, indicating the points of difference. There are various cases, according to the final steps of π_1 and π_2.

1. π_1 is an S-axiom. W.lo.g. we may assume that it is an S_1-axiom. Depending on the Cut-rule, we may replace π as follows.

$$\dfrac{\dfrac{}{\delta(t_1,t_2),\varphi^a_{t_2}\vdash\psi^a_{t_2}}\mathrm{S_1\,ax}\qquad\begin{matrix}\pi_2\\ \psi^a_{t_2},\Gamma'\vdash\Delta'\end{matrix}}{\delta(t_1,t_2),\varphi^a_{t_2},\Gamma'\vdash\Delta'}\mathrm{C}\quad\rightsquigarrow\quad\dfrac{\begin{array}{c}\dfrac{\begin{matrix}\pi_2\\ \psi^a_{t_2},\Gamma'\vdash\Delta'\end{matrix}}{\delta(t_1,t_2),\psi^a_{t_1},\Gamma'\vdash\Delta'}\mathrm{S_2}\\ =\\ \delta(t_1,t_2),\varphi^a_{t_1},\Gamma'\vdash\Delta'\end{array}}{\delta(t_1,t_2),\varphi^a_{t_2},\Gamma'\vdash\Delta'}\mathrm{S_1}$$

$$\dfrac{\dfrac{}{\delta(t_1,t_2),\varphi^a_{t_2}\vdash\psi^a_{t_2}}\mathrm{S_1\,ax}\qquad\begin{matrix}\pi_2\\ \delta(\psi^a_{t_2},t),\Gamma'\vdash\Delta'\end{matrix}}{t,\delta(t_1,t_2),\varphi^a_{t_2},\Gamma'\vdash\Delta'}\mathrm{C\delta L}\quad\rightsquigarrow\quad\dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{matrix}\pi_2\\ \delta(\psi^a_{t_2},t),\Gamma'\vdash\Delta'\end{matrix}}{\psi^a_{t_2},t,\Gamma'\vdash\Delta'}\mathrm{L\delta\uparrow}}{\delta(t_1,t_2),\psi^a_{t_1},t,\Gamma'\vdash\Delta'}\mathrm{S_2}\\ =\\ \delta(t_1,t_2),\varphi^a_{t_1},t,\Gamma'\vdash\Delta'\end{array}}{\delta(t_1,t_2),\varphi^a_{t_2},t,\Gamma'\vdash\Delta'}\mathrm{S_1}$$

$$\dfrac{\dfrac{}{\delta(t_1,t_2),\delta(\varphi^a_{t_2},t^a_{t_2})\vdash\delta(\psi^a_{t_2},u^a_{t_2})}\mathrm{S_1\,ax}\qquad\begin{matrix}\pi_2\\ \psi^a_{t_2},\Gamma'\vdash\Delta'\end{matrix}}{u^a_{t_2},\delta(t_1,t_2),\delta(\varphi^a_{t_2},t^a_{t_2}),\Gamma'\vdash\Delta'}\mathrm{C\delta L}\quad\rightsquigarrow$$

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{}{u^a_{t_2}\vdash u^a_{t_2}}\mathrm{I}}{u^a_{t_2},\Gamma'\vdash\Delta',u^a_{t_2}}\mathrm{W}\qquad\dfrac{\begin{matrix}\pi_2\\ \psi^a_{t_2},\Gamma'\vdash\Delta'\end{matrix}}{\psi^a_{t_2},u^a_{t_2},\Gamma'\vdash\Delta'}\mathrm{W}}{u^a_{t_2},\delta(\psi^a_{t_2},u^a_{t_2}),\Gamma'\vdash\Delta'}\mathrm{L\delta\downarrow}}{u^a_{t_2},\delta(t_1,t_2),\delta(\psi^a_{t_1},u^a_{t_1}),\Gamma'\vdash\Delta'}\mathrm{S_2}\\ =\\ u^a_{t_2},\delta(t_1,t_2),\delta(\varphi^a_{t_1},t^a_{t_1}),\Gamma'\vdash\Delta'\end{array}}{u^a_{t_2},\delta(t_1,t_2),\delta(\varphi^a_{t_2},t^a_{t_2}),\Gamma'\vdash\Delta'}\mathrm{S_1}$$

(We assume that $\varphi^a_{t_1}=\psi^a_{t_1}$ and $t^a_{t_1}=u^a_{t_1}$, as required for the S-axiom, and that π_2 does not contain the parameter a; if it does, we may first replace all occurrences of a by a fresh parameter.)

These transformations give us a proof in LK^*_δ which is strictly less complex than π, and by Claim 1 we may convert it into an S-reduced proof of no greater complexity.

2. π_1 ends in a structural rule: I, W, or one of the Cut-rules. In each case, it is routine to check that this rule commutes with the final cut of π, resulting in a decrease in the height of the proof above the cut of maximal complexity, and so a decrease in the complexity of the proof.
3. π_1 ends in an application of the P-rule. We assume that the parameter a does not occur in π_2; if it does, we must first replace

every occurrence of a in π_1 with a fresh parameter.

$$\dfrac{\dfrac{\begin{matrix}\pi_1'\\ a,\Gamma\vdash\Delta,\varphi_1\end{matrix}}{\Gamma\vdash\Delta,\varphi_1}\,\mathrm{P} \qquad \begin{matrix}\pi_2\\ \varphi_2,\Gamma'\vdash\Delta'\end{matrix}}{\varphi_3,\Gamma,\Gamma'\vdash\Delta,\Delta'} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\pi_1'\\ a,\Gamma\vdash\Delta,\varphi_1\end{matrix} \qquad \begin{matrix}\pi_2\\ \varphi_2,\Gamma'\vdash\Delta'\end{matrix}}{\varphi_3,a,\Gamma,\Gamma'\vdash\Delta,\Delta'}}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathrm{P}$$

4. π_1 ends in an application of one of the logical rules (L$\wedge$, R$\wedge$, L$\vee$, R$\vee$, L$\neg$, R$\neg$, L$\forall$, R$\forall$, L$\exists$, R$\exists$, L$\delta\!\downarrow$, R$\delta\!\downarrow$, L$\delta\!\uparrow$ or R$\delta\!\uparrow$) but φ_1 is *not* the principal formula of that rule. In each case, it is routine to check that each of these rules commute with the final cut of π. As before, the resulting proof is of lower complexity because of the decrease in height of the proofs of the left premises of maximal cuts. We shall illustrate the point with the *L*δ-rule:

$$\dfrac{\dfrac{\begin{matrix}\pi_1'\\ \Gamma\vdash\Delta,\varphi_1,t\end{matrix} \qquad \begin{matrix}\pi_1''\\ \psi,\Gamma\vdash\Delta,\varphi_1\end{matrix}}{\delta(\psi,t),\Gamma\vdash\Delta,\varphi_1}\,\mathrm{L}\delta\!\downarrow \qquad \begin{matrix}\pi_2\\ \varphi_2,\Gamma'\vdash\Delta'\end{matrix}}{\varphi_3,\delta(\psi,t),\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathrm{cut} \quad\rightsquigarrow$$

$$\dfrac{\dfrac{\begin{matrix}\pi_1'\\ \Gamma\vdash\Delta,\varphi_1,t\end{matrix} \qquad \begin{matrix}\pi_2\\ \varphi_2,\Gamma'\vdash\Delta'\end{matrix}}{\varphi_3,\Gamma,\Gamma'\vdash\Delta,\Delta',t}\,\mathrm{cut} \qquad \dfrac{\begin{matrix}\pi_1''\\ \psi,\Gamma\vdash\Delta,\varphi_1\end{matrix} \qquad \begin{matrix}\pi_2\\ \varphi_2,\Gamma'\vdash\Delta'\end{matrix}}{\varphi_3,\psi,\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathrm{cut}}{\varphi_3,\delta(\psi,t),\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathrm{L}\delta\!\downarrow$$

Note that π_1' and π_1'' are both shorter than π_1, and so the two cuts in the resulting proof are both less complex than the cut in π.

5. π_1 ends in an application of the T-rule *and the formulae in* φ_2,Γ,Δ *are all* δ*-formulae*. The cut must be an application of the C-rule ($\varphi_1=\varphi_2=\varphi$, $\varphi_3=\emptyset$) because the other Cut-rules require either φ_1 or φ_2 to be a non-δ-formula. In this case, the T-rule commutes with the cut:

$$\dfrac{\dfrac{\begin{matrix}\pi_1'\\ t,\Gamma\vdash\Delta,\varphi\end{matrix}}{\Gamma\vdash\Delta,\varphi}\,\mathrm{T} \qquad \begin{matrix}\pi_2\\ \varphi,\Gamma'\vdash\Delta'\end{matrix}}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathrm{C} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\pi_1'\\ t,\Gamma\vdash\Delta,\varphi\end{matrix} \qquad \begin{matrix}\pi_2\\ \varphi,\Gamma'\vdash\Delta'\end{matrix}}{t,\Gamma,\Gamma'\vdash\Delta,\Delta'}}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathrm{T}$$

The condition that φ,Γ',Δ' are all δ-formulae is important because if one of these formulae was non-δ then the side-condition on the application of the T-rule in the resulting proof would be invalid.

6. –10. The analogous cases for π_2 instead of π_1.

The above includes all cases except those in which either (i) φ_1 and φ_2 are *both* principal formulae of the final steps of π_1 and π_2 respectively, or (ii) one of the two proofs ends in an application of the T-rule, but the condition specified in 5. (or 10.) does not hold. It is the second, somewhat tricky, case to which we now turn.

11. π_1 ends in an application of the T-rule but some of the formulae in $\varphi_2, \Gamma', \Delta'$ are non-δ-formulae. We consider each of the Cut-rules separately.

    (a) If the cut of π is an application of the CδL-rule then there is a formula ψ and a term u such that $\varphi_1 = \delta(\psi, u)$, $\varphi_2 = \psi$ and $\varphi_3 = u$, and so we are faced with a situation like this:

$$\dfrac{\dfrac{\begin{matrix}\pi_1' \\ t, \Gamma \vdash \Delta, \delta(\psi, u)\end{matrix}}{\Gamma \vdash \Delta, \delta(\psi, u)}\,\text{T} \qquad \begin{matrix}\pi_2 \\ \psi, \Gamma' \vdash \Delta'\end{matrix}}{u, \Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{C}\delta\text{L}$$

By repeated application of the δ-rules we may "protect" the non-δ-formulae of ψ, Γ', Δ' by embedding them inside δ-formulae. Let $\delta(\Gamma', u)$ be the set of formulae $\delta(\gamma, u)$ for each non-δ-formula γ of Γ', together with the δ-formulae of Γ'; and similarly for $\delta(\Delta', u)$. Now replace π by the following proof:

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\pi_1' \\ t, \Gamma \vdash \Delta, \delta(\psi, u)\end{matrix}}{\Gamma \vdash \Delta, \delta(\psi, u)}\,\text{T} \qquad \dfrac{\begin{matrix}\dfrac{\begin{matrix}\pi_2 \\ \psi, \Gamma' \vdash \Delta'\end{matrix}}{u, \psi, \Gamma' \vdash \Delta'}\,\text{W} \\ \vdots \;\; \text{L}\delta\!\downarrow \text{ and R}\delta\!\downarrow \\ u, \delta(\psi, u), \delta(\Gamma', u) \vdash \delta(\Delta', u)\end{matrix}}{\delta(\psi, u), \delta(\Gamma', u) \vdash \delta(\Delta', u)}\,\text{T}}{\Gamma, \delta(\Gamma', u) \vdash \Delta, \delta(\Delta', u)}\,\text{C} \\ \vdots \;\; \text{L}\delta\!\uparrow \text{ and R}\delta\!\uparrow \\ u, \Gamma, \Gamma' \vdash \Delta, \Delta'\end{matrix}$$

Call this proof π'. The maximal cut in π' is actually *more* complex than the maximal cuts in π because its cut-formula is $\delta(\psi, u)$ instead of ψ. However, the right premise of the cut is composed only of δ-formulae. Repeating the argument given cases 1.–5., we see that π' may be simplified until the cut-formula $\delta(\psi, u)$ is the principal formula of the last step in the proof of the left premise. We then have the following situation (possibly repeated many times):

$$\dfrac{\dfrac{\begin{matrix}\pi_0' \\ \Gamma_0 \vdash \Delta_0, u\end{matrix} \qquad \begin{matrix}\pi_0'' \\ \Gamma_0 \vdash \Delta_0, \psi\end{matrix}}{\Gamma_0 \vdash \Delta_0, \delta(\psi, u)}\,\mathrm{R}\delta{\downarrow} \qquad \dfrac{\begin{matrix}\pi_2 \\ \dfrac{\psi, \Gamma' \vdash \Delta'}{u, \psi, \Gamma' \vdash \Delta'}\,\mathrm{W} \\ \vdots\ \ \mathrm{L}\delta{\downarrow} \text{ and } \mathrm{R}\delta{\downarrow} \\ u, \delta(\psi, u), \delta(\Gamma', u) \vdash \delta(\Delta', u)\end{matrix}}{\delta(\psi, u), \delta(\Gamma', u) \vdash \delta(\Delta', u)}\,\mathrm{T}}{\Gamma_0, \delta(\Gamma', u) \vdash \Delta_0, \delta(\Delta', u)}\,\mathrm{C}$$

Each maximal cut in π' will be of this form, and may be replaced by the following:

$$\begin{matrix}\dfrac{\begin{matrix}\pi_0' \\ \Gamma_0 \vdash \Delta_0, u\end{matrix} \qquad \dfrac{\begin{matrix}\pi_0'' \\ \Gamma_0 \vdash \Delta_0, \psi\end{matrix} \qquad \begin{matrix}\pi_2 \\ \psi, \Gamma' \vdash \Delta'\end{matrix}}{\Gamma_0, \Gamma' \vdash \Delta_0, \Delta'}\,\mathrm{C}}{\vdots}\,\mathrm{L}\delta{\downarrow} \text{ and } \mathrm{R}\delta{\downarrow} \\ \Gamma_0, \delta(\Gamma', u) \vdash \Delta_0, \delta(\Delta', u)\end{matrix}$$

Each of these cuts is less complex than the cut in π that we started with, because π_0'' is necessarily shorter than π_1, and so we have reduced the complexity of π, as required.

(b) If the cut of π is an application of the C-rule then $\varphi_1 = \varphi_2 = \varphi$ and $\varphi_3 = \emptyset$, and so we are faced with a situation like this:

$$\dfrac{\dfrac{\begin{matrix}\pi_1' \\ t, \Gamma \vdash \Delta, \varphi\end{matrix}}{\Gamma \vdash \Delta, \varphi}\,\mathrm{T} \qquad \begin{matrix}\pi_2 \\ \varphi, \Gamma' \vdash \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{C}$$

The side-condition on the T-rule requires φ to be a δ-formula, so $\varphi = \delta(\psi, u)$ for some formula ψ and term u. Moreover, we may assume that φ is the principal formula of the final step of π_2; otherwise this case would be subsumed by one of the cases 6.–10. (and if it is case 10. then we may be assured that the extra condition is satisfied, because each of the formulae Γ, Δ, φ is a δ-formula, by the side-condition on the T-rule). Thus we have the following:

$$\dfrac{\dfrac{\begin{matrix}\pi_1'\\ t,\Gamma\vdash\Delta,\delta(\psi,u)\end{matrix}}{\Gamma\vdash\Delta,\delta(\psi,u)}\text{T} \qquad \dfrac{\begin{matrix}\pi_2'\\ \Gamma'\vdash\Delta',u\end{matrix}\qquad\begin{matrix}\pi_2''\\ \psi,\Gamma'\vdash\Delta'\end{matrix}}{\delta(\psi,u),\Gamma'\vdash\Delta'}\text{L}\delta\!\downarrow}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\text{C}$$

which may be modified in the following way. Let a be a fresh parameter.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1'\\ t,\Gamma\vdash\Delta,\delta(\psi,u)\end{matrix}}{\Gamma\vdash\Delta,\delta(\psi,u)}\text{T} \qquad \dfrac{\begin{matrix}\dfrac{\dfrac{\begin{matrix}\pi_2'\\ \Gamma'\vdash\Delta',u\end{matrix}\qquad\begin{matrix}\pi_2''\\ \psi,\Gamma'\vdash\Delta'\end{matrix}}{\delta(\psi,u),\Gamma'\vdash\Delta'}\text{L}\delta\!\downarrow}{a,\delta(\psi,u),\Gamma'\vdash\Delta'}\text{W}\\ \vdots\ \text{L}\delta\!\downarrow\text{ and R}\delta\!\downarrow\\ a,\delta(\psi,u),\delta(\Gamma',a)\vdash\delta(\Delta',a)\end{matrix}}{\delta(\psi,u),\delta(\Gamma',a)\vdash\delta(\Delta',a)}\text{T}}{\begin{matrix}\Gamma,\delta(\Gamma',a)\vdash\Delta,\delta(\Delta',a)\\ \vdots\ \text{L}\delta\!\uparrow\text{ and R}\delta\!\uparrow\\ a,\Gamma,\Gamma'\vdash\Delta,\Delta'\end{matrix}}\text{C}}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\text{P}}{}$$

Again the result is a proof π' which is more complex than π, but which may be converted into a less complex proof by manoeuvres exactly analogous to those made in the previous case.[7]

(c) If the cut of π is an application of the CδR-rule then there is a non-δ-formula ψ and a term u such that $\varphi_1 = \psi$, $\varphi_2 = \delta(\psi, u)$ and $\varphi_3 = u$. This conflicts with the side-condition on the T-rule, which requires φ_1 to be a δ-formula, so this case is void.

12. π_2 ends in an application of the T-rule but some of the formulae in $\varphi_1, \Gamma, \Delta$ are non-δ-formulae. This case is handled in a way directly analogous to case 11.

13. Finally, the only possibility we are left with is that both π_1 and π_2 end in applications of rules whose principal formulae are φ_1 and φ_2 respectively. If the final cut of π is an application of the CδL-rule or CδR-rule then we use the following transformations.

[7] When ψ is not a term, this whole case may be reduced to the previous case.

$$\dfrac{\dfrac{\begin{matrix}\pi_1' \\ \Gamma \vdash \Delta, t\end{matrix} \quad \begin{matrix}\pi_1'' \\ \Gamma \vdash \Delta, \psi\end{matrix}}{\Gamma \vdash \Delta, \delta(\psi, t)}\text{R}\delta\!\downarrow \qquad \begin{matrix}\pi_2 \\ \psi, \Gamma' \vdash \Delta'\end{matrix}}{t, \Gamma, \Gamma' \vdash \Delta, \Delta'}\text{C}\delta\text{L} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\pi_1'' \\ \Gamma \vdash \Delta, \psi\end{matrix} \quad \begin{matrix}\pi_2 \\ \psi, \Gamma' \vdash \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{C}}{t, \Gamma, \Gamma' \vdash \Delta, \Delta'}\text{W}$$

$$\dfrac{\begin{matrix}\pi_1 \\ \Gamma \vdash \Delta, \psi\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2' \\ \Gamma' \vdash \Delta', t\end{matrix} \quad \begin{matrix}\pi_2'' \\ \psi, \Gamma' \vdash \Delta'\end{matrix}}{\delta(\psi, t), \Gamma' \vdash \Delta'}\text{L}\delta\!\downarrow}{t, \Gamma, \Gamma' \vdash \Delta, \Delta'}\text{C}\delta\text{R} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\pi_1 \\ \Gamma \vdash \Delta, \psi\end{matrix} \quad \begin{matrix}\pi_2'' \\ \psi, \Gamma' \vdash \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{C}}{t, \Gamma, \Gamma' \vdash \Delta, \Delta'}\text{W}$$

If the cut of π is an application of the C-rule, we need to examine the various cases.

(a) π_1 ends in an application of the R$\delta\!\uparrow$-rule:

$$\dfrac{\dfrac{\begin{matrix}\pi_1' \\ \Gamma \vdash \Delta, \delta(\varphi, t)\end{matrix}}{t, \Gamma \vdash \Delta, \varphi}\text{R}\delta\!\uparrow \qquad \begin{matrix}\pi_2 \\ \varphi, \Gamma' \vdash \Delta'\end{matrix}}{t, \Gamma, \Gamma' \vdash \Delta, \Delta'}\text{C} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\pi_1' \\ \Gamma \vdash \Delta, \delta(\varphi, t)\end{matrix} \quad \begin{matrix}\pi_2 \\ \varphi, \Gamma' \vdash \Delta'\end{matrix}}{t, \Gamma, \Gamma' \vdash \Delta, \Delta'}\text{C}\delta\text{L}$$

(b) π_2 ends in an application of the L$\delta\!\uparrow$-rule:

$$\dfrac{\begin{matrix}\pi_1 \\ \Gamma \vdash \Delta, \varphi\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2' \\ \delta(\varphi, t), \Gamma' \vdash \Delta'\end{matrix}}{\varphi, t, \Gamma' \vdash \Delta'}\text{L}\delta\!\uparrow}{t, \Gamma, \Gamma' \vdash \Delta, \Delta'}\text{C} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\pi_1 \\ \Gamma \vdash \Delta, \varphi\end{matrix} \quad \begin{matrix}\pi_2' \\ \delta(\varphi, t), \Gamma' \vdash \Delta'\end{matrix}}{t, \Gamma, \Gamma' \vdash \Delta, \Delta'}\text{C}\delta\text{R}$$

(c) π_1 and π_2 end in applications of the right and left rules, respectively, for $\wedge$, $\vee$, $\neg$, $\exists$ or $\forall$: use the classical transformations.

(d) π_1 ends in an application of R$\delta\!\downarrow$ and π_2 ends in an application of L$\delta\!\downarrow$:

$$\dfrac{\dfrac{\begin{matrix}\pi_1' \\ \Gamma \vdash \Delta, t\end{matrix} \quad \begin{matrix}\pi_1'' \\ \Gamma \vdash \Delta, \psi\end{matrix}}{\Gamma \vdash \Delta, \delta(\psi, t)}\text{R}\delta\!\downarrow \quad \dfrac{\begin{matrix}\pi_2' \\ \Gamma' \vdash \Delta', t\end{matrix} \quad \begin{matrix}\pi_2'' \\ \psi, \Gamma' \vdash \Delta'\end{matrix}}{\delta(\psi, t), \Gamma' \vdash \Delta'}\text{L}\delta\!\downarrow}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{C} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\pi_1'' \\ \Gamma \vdash \Delta, \psi\end{matrix} \quad \begin{matrix}\pi_2'' \\ \psi, \Gamma' \vdash \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{C}$$

This completes the proof of Claim 2.

The theorem may now be proved. Each proof in LK_δ^* may be converted into an S-reduced proof of no greater complexity, by Claim 1.

This proof has a finite number of cuts of maximal complexity. Claim 2 shows how a sub-proof ending in each one these maximal cuts may be replaced by a less complex proof. Apply this operation to each of the maximal cuts, and the resulting proof will be strictly less complex than the original. Thus, by induction on the complexity of proofs, we see that every LK_δ^* may be converted into a proof of minimal complexity. The only LK_δ^* proofs of minimal complexity are those of LK_δ augmented with S-axioms. Finally, the S-axioms are obviously derivable in LK_δ, using the S-rules, and so the theorem is proved.