# INTERPOLATION FOR EXTENDED MODAL LANGUAGES

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**Abstract.** Several extensions of the basic modal language are characterized in terms of interpolation. Our main results are of the following form: *Language $\mathcal{L}'$ is the least expressive extension of $\mathcal{L}$ with interpolation.* For instance, let $\mathcal{M}(\mathsf{D})$ be the extension of the basic modal language with a difference operator [7]. First-order logic is the least expressive extension of $\mathcal{M}(\mathsf{D})$ with interpolation. These characterizations are subsequently used to derive new results about hybrid logic, relation algebra and the guarded fragment.

**§1. Introduction.** In this paper, we consider extensions of the basic modal language that involve reference to individual states of a Kripke structure. A typical example is the language $\mathcal{H}(\mathsf{E})$, in which one can refer to individual states of the Kripke model using nominals (similar to constants in first-order logic) and the universal modality [9]. Another example is *difference logic* $\mathcal{M}(\mathsf{D})$, i.e., the extension of the basic modal language with an extra operator $\mathsf{D}$ such that $\mathfrak{M}, w \models \mathsf{D}\phi$ iff $\mathfrak{M}, v \models \phi$ for some $v \neq w$ [7].

$\mathcal{M}(\mathsf{D})$ and $\mathcal{H}(\mathsf{E})$ are known not to have interpolation [5]. In this paper, we systematically investigate extensions of these languages, trying to restore interpolation. We show that, in a precise sense, first-order logic is the smallest extension of these languages with interpolation. This can be seen as a characterization of first-order logic or, from another perspective, as a general negative interpolation result. In a similar way, we characterize the hybrid language $\mathcal{H}(@, \downarrow)$ as the smallest extension of $\mathcal{H}(@)$ with interpolation [2].

These characterizations are subsequently used to derive new results concerning interpolation for hybrid logic, relation algebra and the guarded fragment. We prove as a corollary that there is no decidable hybrid language with Craig interpolation, thus answering an open question raised by Areces, Blackburn, and Marx [2].

**§2. Abstract Modal and Hybrid Languages.** Before we can state general results saying that one language is the smallest extension of another language with interpolation, we need to give an abstract definition of languages. In the

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present section, we give such a definition. In fact, we define two types of languages: *hybrid languages*, relative to a signature that contains both propositional variables and nominals, and *modal languages*, relative to a signature that consists only of propositional variables.

Throughout this paper, we will assume a fixed set of modalities MOD. Each modality $\triangle \in \text{MOD}$ has an associated arity $n(\triangle)$. A modality of arity $n$ is semantically interpreted by an $n+1$-ary relation. More precisely, a *frame* is a structure $\mathfrak{F} = (W, R_\triangle)_{\triangle \in \text{MOD}}$, where $W$ is a non-empty set of worlds and $R_\triangle \subseteq W^{n(\triangle)+1}$ for each $\triangle \in \text{MOD}$.

A hybrid signature is a pair $\sigma = (\text{PROP}_\sigma, \text{NOM}_\sigma)$ of disjoint sets containing propositional variables and nominals respectively. Nominals are similar like propositional variables, except for the fact that their denotation in a model is always a singleton set (i.e, nominals denote points rather than sets). We will often be sloppy by using $\sigma$ to denote the union $\text{PROP}_\sigma \cup \text{NOM}_\sigma$. For instance, we will write $\sigma \subseteq \tau$ instead of $\text{PROP}_\sigma \subseteq \text{PROP}_\tau$ & $\text{NOM}_\sigma \subseteq \text{NOM}_\tau$. A modal signature is a hybrid signature that contains no nominals.

Given a signature $\sigma$, a (pointed, but not necessarily point-generated) *$\sigma$-model* is a structure $\mathfrak{M} = (\mathfrak{F}, V, w)$ where $\mathfrak{F} = (W, R_\triangle)_{\triangle \in \text{MOD}}$ is a frame, $V : \text{PROP}_\sigma \cup \text{NOM}_\sigma \to \wp(W)$ a valuation and $w \in W$ a world, such that $|V(i)| = 1$ for all $i \in \text{NOM}_\sigma$. Observe that for modal signatures, these structures are plain Kripke models. The class of all $\sigma$-models is denoted by $\text{Str}[\sigma]$. Furthermore, for any class of frames $\mathsf{F}$, $\text{Str}_\mathsf{F}[\sigma]$ will denote the class of $\sigma$-models of which the underlying frame belongs to $\mathsf{F}$.

Two operations on models will be useful later on. Firstly, a renaming $\rho : \sigma \to \tau$ is a mapping from $\sigma$ to $\tau$ that respects the sorting: it maps elements of $\text{PROP}_\sigma$ to elements of $\text{PROP}_\tau$ and elements of $\text{NOM}_\sigma$ to elements of $\text{NOM}_\tau$. For any model $\mathfrak{M} = (\mathfrak{F}, V, w) \in \text{Str}[\tau]$ and renaming $\rho : \sigma \to \tau$, let $\mathfrak{M}^\rho$ be the $\sigma$-model $(\mathfrak{F}, \rho \cdot V, w)$. Secondly, if $\mathfrak{M} \in \text{Str}[\tau]$ and $\sigma \subseteq \tau$, then $\mathfrak{M} \upharpoonright \sigma$ denotes the $\sigma$-reduct of $\mathfrak{M}$, i.e., the $\sigma$-model that is obtained from $\mathfrak{M}$ by "forgetting" the interpretation of $\tau \backslash \sigma$. We write $\mathsf{K} \upharpoonright \sigma$ for $\{\mathfrak{M} \upharpoonright \sigma \mid \mathfrak{M} \in \mathsf{K}\}$.

DEFINITION 2.1 (Modal and hybrid languages). *A* modal (hybrid) language *is a pair $(\mathcal{L}, \models_\mathcal{L})$, where $\mathcal{L}$ is a map from modal (hybrid) signatures to sets of formulas, and $\models_\mathcal{L}$ is a relation between formulas and models satisfying the following conditions.*

1. **Expansion Property.** *If $\sigma \subseteq \tau$ then $\mathcal{L}[\sigma] \subseteq \mathcal{L}[\tau]$. Furthermore, for all $\phi \in \mathcal{L}[\sigma]$ and $\mathfrak{M} \in \text{Str}[\tau]$, $\mathfrak{M} \models_\mathcal{L} \phi$ iff $\mathfrak{M} \upharpoonright \sigma \models_\mathcal{L} \phi$. If $\mathfrak{M} \in \text{Str}[\sigma]$ and $\mathfrak{M} \models \phi$, then $\phi \in \mathcal{L}[\sigma]$.*
2. **Renaming Property** *For all $\phi \in \mathcal{L}[\sigma]$ and renamings $\rho : \sigma \to \tau$, there is a $\psi \in \mathcal{L}[\tau]$ such that for all $\mathfrak{M} \in \text{Str}[\tau]$, $\mathfrak{M} \models \psi$ iff $\mathfrak{M}^\rho \models \phi$.*

Definition 2.1 is inspired by similar ones occurring in the literature on abstract model theory [3]. Since the definition is rather general, one might ask what is still *modal* about these abstract modal languages. The two main distinctively modal features in Definition 2.1 are (1) the fact that that the structures we work with are pointed, reflecting the fact that modal formulas are always evaluated locally, and (2) the strict distinction between modalities on the one hand and propositional variables and nominals on the other hand. The importance of this distinction will become clear later on, when we'll consider specific classes of frames.

Some shorthand notation will be convenient. Firstly, by a slight abuse of notation, we will often use $\mathcal{L}$ to refer to the pair $(\mathcal{L}, \models_{\mathcal{L}})$. Secondly, given a model $\mathfrak{M} = (\mathfrak{F}, V, w)$ and an element $v$ of the frame $\mathfrak{F}$, we will use $(\mathfrak{M}, v)$ to denote the model $(\mathfrak{F}, V, v)$. Thus, with $\mathfrak{M}, v \models \phi$ we mean $(\mathfrak{F}, V, v) \models \phi$. Next, for $\phi \in \mathcal{L}[\sigma]$, let $\mathrm{Mod}^{\sigma}_{\mathcal{L}}(\phi) = \{\mathfrak{M} \in \mathrm{Str}[\sigma] \mid \mathfrak{M} \models_{\mathcal{L}} \phi\}$. For $\mathfrak{M} \in \mathrm{Str}[\sigma]$ and $\phi \in \mathcal{L}[\sigma]$, let $[\![\phi]\!]^{\mathfrak{M}}_{\mathcal{L}} = \{v \mid \mathfrak{M}, v \models \phi\}$, i.e., the subset of the domain of $\mathfrak{M}$ defined by $\phi$. Finally, the symbol $\models$ will be used not only to refer to the satisfaction relation, but also to the *local consequence* relation: for $\Phi \cup \{\psi\} \subseteq \mathcal{L}[\sigma]$, we say that $\Phi \models_{\mathcal{L}} \phi$ iff for all $\mathfrak{M} \in Str[\sigma]$, it holds that if $\mathfrak{M} \models_{\mathcal{L}} \phi$ for $\phi \in \Phi$ then $\mathfrak{M} \models_{\mathcal{L}} \psi$.

Often, we will restrict attention to a specific frame class $\mathsf{F}$. In these cases, we will write $\mathrm{Mod}^{\sigma}_{\mathcal{L},\mathsf{F}}(\phi)$ for $\{\mathfrak{M} \in \mathrm{Str}_{\mathsf{F}}[\sigma] \mid \mathfrak{M} \models_{\mathcal{L}} \phi\}$. Likewise, for $\Phi \cup \{\psi\} \subseteq \mathcal{L}[\sigma]$, we say that $\Phi \models_{\mathcal{L},\mathsf{F}} \phi$ iff $\bigcap_{\phi\in\Phi} \mathrm{Mod}^{\sigma}_{\mathcal{L},\mathsf{F}}(\phi) \subseteq \mathrm{Mod}^{\sigma}_{\mathcal{L},\mathsf{F}}(\psi)$.

Definition 2.2 (Extensions of modal or hybrid languages). *Let $\mathcal{L}, \mathcal{L}'$ be modal (hybrid) languages. Then $\mathcal{L}'$ extends $\mathcal{L}$ relative to a frame class $\mathsf{F}$ (notation: $\mathcal{L} \subseteq_{\mathsf{F}} \mathcal{L}'$) if the following holds for all modal (hybrid) signatures $\sigma$ and propositional variables $p_1, \ldots, p_n$ ($n \geq 0$).*

- *For each $\phi \in \mathcal{L}[\sigma \cup \{p_1, \ldots, p_n\}]$ and $\psi_1, \ldots, \psi_n \in \mathcal{L}'[\sigma]$, there is a formula of $\mathcal{L}'[\sigma]$, which we will denote by $\phi^{[\vec{p}/\vec{\psi}]}$, such that for all $\mathfrak{M} \in \mathrm{Str}_{\mathsf{F}}[\sigma]$, $\mathfrak{M} \models_{\mathcal{L}'} \phi^{[\vec{p}/\vec{\psi}]}$ iff $\mathfrak{M}^{[p_1 \mapsto [\![\psi_1]\!]^{\mathfrak{M}}_{\mathcal{L}'}, \ldots, p_n \mapsto [\![\psi_n]\!]^{\mathfrak{M}}_{\mathcal{L}'}]} \models_{\mathcal{L}} \phi$.*

Note that Definition 2.2 concerns *expressive* extensions rather than *axiomatic* extensions. As a special case (take $n = 0$), we have that whenever $\mathcal{L} \subseteq_{\mathsf{F}} \mathcal{L}'$ and $\phi \in \mathcal{L}[\sigma]$, there is a $\psi \in \mathcal{L}'[\sigma]$ such that $\mathrm{Mod}^{\sigma}_{\mathcal{L},\mathsf{F}}(\phi) = \mathrm{Mod}^{\sigma}_{\mathcal{L}',\mathsf{F}}(\psi)$. However, Definition 2.2 provides more information: it ensures that $\mathcal{L}'$ is closed under the basic operations of $\mathcal{L}$, such as negation. For, suppose $\mathcal{L} \subseteq_{\mathsf{F}} \mathcal{L}'$ and $\mathcal{L}$ has negation. Then for any $\phi \in \mathcal{L}'$, $(\neg p)^{[p/\phi]}$ expresses the negation of $\phi$. Definitions like Definition 2.2 are quite common in the literature on abstract model theory. Incidentally, such definitions makes sense only for languages $\mathcal{L}$ that are closed under substitution of formulas for propositional variables, since otherwise it might happen that $\mathcal{L} \not\subseteq \mathcal{L}$. All languages that we will be concerned with in this paper are closed under substitution.

Finally, let us define interpolation. In the context of modal logic, interpolation can be defined in several ways. The following definition captures what is often called *local interpolation* or *arrow interpolation* (keep in mind that $\models_{\mathcal{L}}$ is the *local* consequence relation).

Definition 2.3 (Interpolation). *A modal or hybrid language $\mathcal{L}$ has* interpolation *on a frame class $\mathsf{F}$ if for all $\phi \in \mathcal{L}[\sigma]$ and $\psi \in \mathcal{L}[\tau]$ such that $\phi \models_{\mathcal{L},\mathsf{F}} \psi$, there is a $\theta \in \mathcal{L}[\sigma \cap \tau]$ such that $\phi \models_{\mathcal{L},\mathsf{F}} \theta$, and $\theta \models_{\mathcal{L},\mathsf{F}} \psi$.*

Note that, since the modalities are not part of the signature, no requirements are made on the modalities occuring in the interpolant $\theta$. The nominals occurring in $\theta$, however, must occur both in $\phi$ and in $\psi$.

**§3. Examples of modal and hybrid languages.** The basic modal language $\mathcal{BML}$ [4] is the prototypical example of a modal language in the sense of Definition 2.1. In this section, we list a number of other modal and hybrid languages. Keep in mind that we make no assumptions on the set of modalities MOD, and that modality $\triangle \in$ MOD has an associated arity $n(\triangle)$.

- The language of difference logic, which we will denote by $\mathcal{M}(\mathsf{D})$, is the extension of the basic modal language with an extra operator $\mathsf{D}$, such that $\mathfrak{M}, w \models \mathsf{D}\phi$ iff $\mathfrak{M}, v \models \phi$ for some world $v \neq w$ [7]. Difference logic is a modal language.
- The basic hybrid language $\mathcal{H}(@)$ is obtained by extending the basic modal language with nominals $i, j, k, \ldots$ and with a satisfaction operator $@_i$ for each nominal $i$. For example, the formula $\Box\Diamond i$ holds at a point $w$ just in case every successor of $w$ can see the unique point at which the nominal $i$ is true, and $@_i p$ holds just in case $p$ is true at the point named by the nominal $i$. More precisely, given a signature $\sigma = (\text{PROP}, \text{NOM})$, the formulas of $\mathcal{H}(@)[\sigma]$ are given by $\phi ::= p \mid i \mid \top \mid \phi \wedge \psi \mid \neg\phi \mid \triangle(\phi_1, \ldots, \phi_{n(\triangle)}) \mid @_i\phi$ where $p \in$ PROP, $i \in$ NOM and $\triangle \in$ MOD. For any model $\mathfrak{M} = (W, R, V, w) \in \text{Str}[\sigma]$, $\mathfrak{M} \models i$ iff $V(i) = \{w\}$ and $\mathfrak{M} \models @_i\phi$ iff $\mathfrak{M}, v \models \phi$ where $V(i) = \{v\}$.
- The slightly more expressive hybrid language $\mathcal{H}(\mathsf{E})$ is obtained by the basic modal language with nominals and the global modality $\mathsf{E}$. Formally, the formulas of $\mathcal{H}(\mathsf{E})$ are given by $\phi ::= p \mid i \mid \neg\phi \mid \phi \wedge \psi \mid \triangle(\phi_1, \ldots, \phi_{n(\triangle)}) \mid \mathsf{E}\phi$, where $p \in$ PROP, $i \in$ NOM and $\triangle \in$ MOD. The truth definition is such that $\mathfrak{M}, w \models \mathsf{E}\phi$ iff $\mathfrak{M}, v \models \phi$ for some world $v$ of $\mathfrak{M}$. Notice that the satisfaction operators are definable in terms of the global modality: $@_i\phi$ is equivalent to $\mathsf{E}(i \wedge \phi)$.
- Another very expressive hybrid language $\mathcal{H}(@, \downarrow)$ is obtained by extending $\mathcal{H}(@)$ with the $\downarrow$-binder, which allows explicit reference to the current point of evaluation. For example, the formula $\downarrow x.\Diamond x$ expresses that the current world is a successor of itself. Formally, let VAR be a countably infinite set of variables, and for any formula $\phi$, nominal $i$ and variable $x$, let $\phi[i/x]$ be the result of replacing all occurrences of $i$ in $\phi$ by $x$. Then the formulas of $\mathcal{H}(@, \downarrow)$ are given by $\phi ::= p \mid i \mid \neg\phi \mid \phi \wedge \psi \mid \triangle(\phi_1, \ldots, \phi_{n(\triangle)}) \mid @_i\phi \mid \downarrow x.\phi[i/x]$, where $p \in$ PROP, $i \in$ NOM, $\triangle \in$ MOD and $x \in$ VAR, such that $x$ is substitutable for $i$ in $\phi$. Note that this definition uses a trick familiar from first-order logic to avoid formulas with free variables. The truth definition is such that $\mathfrak{M}, w \models \downarrow x.\phi[i/x]$ iff $\mathfrak{M}^{[i \mapsto \{w\}]}, w \models \phi$, where the model $\mathfrak{M}^{[i \mapsto \{w\}]}$ is identical to $\mathfrak{M}$ except that $i$ denotes $\{w\}$.
- First-order logic also constitutes a hybrid language, in the following sense. For every signature $\sigma = (\text{PROP}, \text{NOM})$, let $\sigma^*$ be the first-order logic signature that has PROP as its unary predicates, NOM as its constants, and that has a relation $R_\triangle$ of arity $n(\triangle) + 1$ for each $\triangle \in$ MOD. Fix a first-order variable $x$, and for all signatures $\sigma$, let $\mathcal{L}^1[\sigma]$ be collection of first-order formulas with no free variables besides $x$, in the signature $\sigma^*$ with equality. Furthermore, let $\mathfrak{M}, w \models_{\mathcal{L}^1} \phi(x)$ if $\phi(x)$ holds in $\mathfrak{M}$ conceived of as an ordinary first-order structure, interpreting $x$ as $w$. Then $(\mathcal{L}^1, \models_{\mathcal{L}^1})$ is a hybrid

| | | | | | |
|---|---|---|---|---|---|
| modal languages | $\mathcal{BML}$ **yes** | $\subseteq$ | $\mathcal{M}(\mathsf{D})$ no | $\subseteq$ | $\mathcal{L}^1$ **yes** |

hybrid languages:

$\mathcal{H}(@)$ no $\subsetneq$ $\mathcal{H}(\mathsf{E})$ no $\subsetneq$ $\mathcal{L}^1$ **yes**

$\mathcal{H}(@)$ no $\subsetneq$ $\mathcal{H}(@, \downarrow)$ **yes** $\subsetneq$ $\mathcal{L}^1$ **yes**

FIGURE 1. Modal and hybrid languages ordered by expressivity.

language, and we will refer to it as *the first-order correspondence language*, or simply *first-order logic*.

Similarly, we can view first-order logic as a *modal* language, by restricting attention to signatures that contain no nominals. In what follows, we will not distinguish $(\mathcal{L}^1, \models_{\mathcal{L}^1})$ as a hybrid language from $(\mathcal{L}^1, \models_{\mathcal{L}^1})$ as a modal language. It will be clear from the context which we are referring to.

Definition 2.1 covers many other languages, including infinitary and higher-order ones. However, the above mentioned languages will play an important role in the next section. Definition 2.2 orders these modal languages into the hierarchy depicted in Figure 1. Assuming we have at least one unary modality and that we work with the class of all frames, all inclusions indicated in Figure 1 are strict. It is also indicated which languages have interpolation: **yes** indicates that a language has interpolation on the class of all frames, and **no** indicates that it does not. All these results are known, cf. [7, 9, 2, 6, 8, 5].

**§4. Main results.** The following results, which form the main contribution of this paper, complement the known results depicted in Figure 1. They show that every language between $\mathcal{H}(@)$ and $\mathcal{H}(@, \downarrow)$, between $\mathcal{H}(\mathsf{E})$ and $\mathcal{L}^1$ or between $\mathcal{M}(\mathsf{D})$ and $\mathcal{L}^1$ lacks interpolation. In fact, this holds relative to any frame class $\mathsf{F}$.

THEOREM 4.1. *Let* $\mathsf{F}$ *be any class of frames (we make no assumptions on the collection of modalities* MOD*). Then the following hold.*[1]

1. *For all hybrid languages* $\mathcal{L}$*, if* $\mathcal{H}(@) \subseteq_{\mathsf{F}} \mathcal{L}$ *and* $\mathcal{L}$ *has interpolation on* $\mathsf{F}$ *then* $\mathcal{H}(@, \downarrow) \subseteq_{\mathsf{F}} \mathcal{L}$
2. *For all hybrid languages* $\mathcal{L}$*, if* $\mathcal{H}(\mathsf{E}) \subseteq_{\mathsf{F}} \mathcal{L}$ *and* $\mathcal{L}$ *has interpolation on* $\mathsf{F}$ *then* $\mathcal{L}^1 \subseteq_{\mathsf{F}} \mathcal{L}$

[1] Recall that we use the notation $\mathcal{L}^1$ ambiguously. In the second clause of the theorem, we refer to it as a hybrid language, whereas in the third clause of the theorem, we refer to it as a modal language.

3. *For all modal languages* $\mathcal{L}$*, if* $\mathcal{M}(\mathsf{D}) \subseteq_\mathsf{F} \mathcal{L}$ *and* $\mathcal{L}$ *has interpolation on* $\mathsf{F}$ *then* $\mathcal{L}^1 \subseteq_\mathsf{F} \mathcal{L}$

These results can be interpreted as general negative interpolation results, or, from another perspective, as characterizations. For instance, Theorem 4.1.1 characterizes $\mathcal{H}(@, \downarrow)$ as the smallest extension of $\mathcal{H}(@)$ that has interpolation.

The remainder of this section is devoted to the proof of Theorem 4.1. First, we prove an adapted version of well-known lemma relating interpolation with *projective classes* [3].

Definition 4.1 (Projective classes). *Let* $\sigma$ *be a modal (hybrid) signature, and let* $\mathsf{K} \subseteq \mathrm{Str}_\mathsf{F}[\sigma]$*. Then* $\mathsf{K}$ *is a* projective class *of a modal (hybrid) language* $\mathcal{L}$ *relative to a frame class* $\mathsf{F}$ *if there is a* $\phi \in \mathcal{L}[\tau]$ *with* $\tau \supseteq \sigma$ *a modal (hybrid) signature, such that* $\mathsf{K} = \mathrm{Mod}^\tau_{\mathcal{L},\mathsf{F}}(\phi) \restriction \sigma$*.*

Definition 4.2 (Negation). *A modal or hybrid language* $\mathcal{L}$ has negation *on* $\mathsf{F}$ *if for each* $\phi \in \mathcal{L}[\sigma]$ *there is an formula of* $\mathcal{L}[\sigma]$*, which we will denote by* $\neg\phi$*, such that* $\mathrm{Mod}_{\mathcal{L},\mathsf{F}}(\psi) = \mathrm{Str}_\mathsf{F}[\sigma] \backslash \mathrm{Mod}_{\mathcal{L},\mathsf{F}}(\phi)$*.*

Lemma 4.1. *Let* $\mathcal{L}$ *be a modal (hybrid) language with negation that has interpolation on a frame class* $\mathsf{F}$*, and let* $\mathsf{K} \subseteq \mathrm{Str}_\mathsf{F}[\sigma]$*, for some modal (hybrid) signature* $\sigma$*. If both* $\mathsf{K}$ *and* $\mathrm{Str}_\mathsf{F}[\sigma]\backslash\mathsf{K}$ *are projective classes of* $\mathcal{L}$ *relative to* $\mathsf{F}$*, then there is a* $\phi \in \mathcal{L}[\sigma]$ *such that* $\mathsf{K} = \mathrm{Mod}_{\mathcal{L},\mathsf{F}}(\phi)$*.*

Proof. Since $\mathsf{K}$ is a projective class, there is a formula $\phi \in \mathcal{L}[\sigma \cup \tau]$ such that $\mathsf{K} = \mathrm{Mod}_{\mathcal{L},sF}(\phi) \restriction \sigma$. Likewise, since $\mathrm{Str}_\mathsf{F}[\sigma]\backslash\mathsf{K}$ is a projective class, there is a formula $\psi \in \mathcal{L}[\sigma \cup \tau']$ such that $\mathrm{Str}_\mathsf{F}[\sigma]\backslash\mathsf{K} = \mathrm{Mod}_{\mathcal{L},\mathsf{F}}(\psi) \restriction \sigma$. Without loss of generality, we can assume that $\tau$ and $\tau'$ are disjoint (by the Renaming property of $\mathcal{L}$). It follows that $\phi \models_{\mathcal{L},\mathsf{F}} \neg\psi$. Since $\mathcal{L}$ has interpolation, there must be a $\theta \in \mathcal{L}[\sigma]$ such that $\phi \models_{\mathcal{L},\mathsf{F}} \theta$ and $\theta \models_{\mathcal{L},\mathsf{F}} \neg\psi$. As a last step, we will show that $\mathrm{Mod}_{\mathcal{L},\mathsf{F}}(\theta) = \mathsf{K}$.

Suppose $\mathfrak{M} \in \mathsf{K}$. Then $\mathfrak{M} = \mathfrak{N} \restriction \sigma$ for some $\mathfrak{N} \in \mathrm{Mod}_{\mathcal{L},\mathsf{F}}(\phi)$. Since $\phi \models_{\mathcal{L},\mathsf{F}} \theta$, it follows that $\mathfrak{N} \models \theta$. By the Expansion property, $\mathfrak{M} \models \theta$. Conversely, suppose $\mathfrak{M} \notin \mathsf{K}$. Then $\mathfrak{M} = \mathfrak{N} \restriction \sigma$ for some $\mathfrak{N} \in \mathrm{Mod}_{\mathcal{L},\mathsf{F}}(\psi)$. Since $\theta \models_{\mathcal{L},\mathsf{F}} \neg\psi$, it follows that $\mathfrak{N} \not\models \theta$. By the Expansion property, $\mathfrak{M} \not\models \theta$. ⊣

The property expressed in Lemma 4.1 is called $\Delta$-interpolation [3]. It is a slightly weaker condition than interpolation, but it arguably more natural from a model theoretic perspective. Incidentally, it should be mentioned that Theorem 4.1 may be strengthened by replacing the condition of interpolation by that of $\Delta$-interpolation.

Using Lemma 4.1, we can show that if the $\downarrow$-binder is added to a hybrid language with interpolation extending $\mathcal{H}(@)$, or to a modal language with interpolation extending $\mathcal{M}(\mathsf{D})$, then the expressivity of the language in question does not increase. This is expressed in the following two lemmas.

Lemma 4.2. *Let* $\mathcal{L}$ *be a hybrid language with interpolation on* $\mathsf{F}$ *such that* $\mathcal{H}(@) \subseteq_\mathsf{F} \mathcal{L}$*. For all* $\phi \in \mathcal{L}[\sigma]$ *and* $i \in \textsc{nom}_\sigma$*, there is a formula of* $\mathcal{L}[\sigma\backslash\{i\}]$*, which we will denote by* $\downarrow i.\phi$*, such that* $\mathrm{Mod}_{\mathcal{L},\mathsf{F}}(\downarrow i.\phi) = \{(\mathfrak{F}, V, w) \in \mathrm{Str}_\mathsf{F}[\sigma\backslash\{i\}] \mid (\mathfrak{F}, V^{[i \mapsto \{w\}]}, w) \models \phi\}$

PROOF. Let $\mathsf{K}_{\downarrow i.\phi} = \{(\mathfrak{F}, V, w) \in \mathrm{Str}_{\mathsf{F}}[\sigma\backslash\{i\}] \mid (\mathfrak{F}, V^{[i\mapsto\{w\}]}, w) \models \phi\}$. $\mathsf{K}_{\downarrow i.\phi}$ is projectively defined by $i \wedge \phi$ and its complement is projectively defined by $i \wedge \neg\phi$. Since $\mathcal{L}$ has negation and has interpolation on $\mathsf{F}$, by Lemma 4.1 $\mathsf{K}_{\downarrow i.\phi} = \mathrm{Mod}_{\mathcal{L},\mathsf{F}}(\psi)$ for some $\psi \in \mathcal{L}[\sigma\backslash\{i\}]$. ⊣

LEMMA 4.3. *Let $\mathcal{L}$ be a modal language with interpolation on $\mathsf{F}$ such that $\mathcal{M}(\mathsf{D}) \subseteq_{\mathsf{F}} L$. For all $\phi \in \mathcal{L}[\sigma]$ and $p \in \mathrm{PROP}_\sigma$, there is a formula of $\mathcal{L}[\sigma\backslash\{p\}]$, which we shall denote by $\downarrow p.\phi$, such that* $\mathrm{Mod}_{\mathcal{L},\mathsf{F}}(\downarrow p.\phi) = \{(\mathfrak{F}, V, w) \in \mathrm{Str}_{\mathsf{F}}[\sigma\backslash\{p\}] \mid (\mathfrak{F}, V^{[p\mapsto\{w\}]}, w) \models \phi\}$

PROOF. Let $\mathsf{K}_{\downarrow p.\phi} = \{(\mathfrak{F}, V, w) \in \mathrm{Str}_{\mathsf{F}}[\sigma\backslash\{p\}] \mid (\mathfrak{F}, V^{[p\mapsto\{w\}]}, w) \models \phi\}$. $\mathsf{K}_{\downarrow p.\phi}$ is projectively defined by $p \wedge \neg\mathsf{D}p \wedge \phi$ and its complement is projectively defined by $p \wedge \neg\mathsf{D}p \wedge \neg\phi$. Since $\mathcal{L}$ has negation and has interpolation on $\mathsf{F}$, by Lemma 4.1 $\mathsf{K}_{\downarrow p.\phi} = \mathrm{Mod}_{\mathcal{L},\mathsf{F}}(\psi)$ for some $\psi \in \mathcal{L}[\sigma\backslash\{p\}]$. ⊣

We are now ready to prove Theorems 4.1.1– 4.1.3.

PROOF OF THEOREM 4.1.1 . Let $\mathcal{L}$ be any hybrid language with interpolation on $\mathsf{F}$ such that $\mathcal{H}(@) \subseteq_{\mathsf{F}} \mathcal{L}$. Let $\phi \in \mathcal{H}(@,\downarrow)[\sigma \cup \{p_1, \dots, p_n\}]$ and $\psi_1, \dots, \psi_n \in \mathcal{L}[\sigma]$. We will show that there is a formula $\chi \in \mathcal{L}[\sigma]$ that is $[\vec{p}/\vec{\psi}]$-equivalent to $\phi$ on $\mathsf{F}$, meaning that

$$\text{for all } \mathfrak{M} \in \mathrm{Str}_{\mathsf{F}}[\sigma],\ \mathfrak{M} \models_{\mathcal{L}} \chi \text{ iff } \mathfrak{M}^{[p_1\mapsto[\![\psi_1]\!]^{\mathfrak{M}}_{\mathcal{L}}, \dots, p_n\mapsto[\![\psi_n]\!]^{\mathfrak{M}}_{\mathcal{L}}]} \models_{\mathcal{H}(@)} \phi$$

The proof proceeds by induction on the length of $\phi$. The base case (where $\phi$ is a propositional variable or nominal from $\sigma$, or $\phi$ is $\top$ or $\phi$ is $p_i$ for some $i \le n$) follows immediately from the fact that $\mathcal{H}(@) \subseteq_{\mathsf{F}} \mathcal{L}$. For the inductive step, we will only prove the cases for negation and for the $\downarrow$-binder, since the other cases are similar to the one for negation.

Let $\phi$ be of the form $\neg\psi$. By induction hypothesis, $\psi$ is $[\vec{p}/\vec{\psi}]$-equivalent on $\mathsf{F}$ to some $\chi \in \mathcal{L}[\sigma]$. Let $q$ be any propositional variable not in $\sigma$ and distinct from $p_1, \dots, p_n$. Since $\mathcal{H}(@) \subseteq_{\mathsf{F}} \mathcal{L}$ and $(\neg q) \in \mathcal{H}(@)[\sigma \cup \{q\}]$, Definition 2.2 guarantees the existence of a formula $(\neg p)^{[p/\chi]} \in \mathcal{L}[\sigma]$ that expresses the negation of $\psi$ on $\mathsf{F}$. It follows that $(\neg p)^{[p/\chi]}$ is $[\vec{p}/\vec{\psi}]$-equivalent on $\mathsf{F}$ to $\phi$.

Let $\phi$ be of the form $\downarrow x.\psi$. Let $i$ be any nominal not in $\sigma$. By the induction hypothesis, we know that there is some $\chi \in \mathcal{L}[\sigma \cup \{i\}]$ that is $[\vec{p}/\vec{\psi}]$-equivalent on $\mathsf{F}$ to $\psi[x/i]$. By Lemma 4.2 it follows that $\downarrow x.\psi$ is $[\vec{p}/\vec{\psi}]$-equivalent on $\mathsf{F}$ to $\downarrow i.\chi \in \mathcal{L}[\sigma]$. ⊣

PROOF OF THEOREM 4.1.2 . Similar to the proof of Theorem 4.1.1. We will only discuss the inductive step for formulas of the form $\exists y.\psi$.

Let $\phi \in \mathcal{L}^1[\sigma]$ be of the form $\exists y.\psi$. By the definition of $\mathcal{L}^1$, $\phi$ contains at most one free variable, say $x$ (in case $\phi$ contains no free variables, let $x$ be any variable distinct from $y$). Let $i, j$ be distinct nominals (constants) not in $\sigma$. By induction hypothesis, $\phi[x/i, y/j] \in \mathcal{L}^1[\sigma \cup \{i, j\}]$ is $[\vec{p}/\vec{\psi}]$-equivalent on $\mathsf{F}$ to some $\chi \in \mathcal{L}[\sigma \cup \{i, j\}]$. By Lemma 4.2 and by the fact that $\mathcal{H}(\mathsf{E}) \subseteq_{\mathsf{F}} \mathcal{L}$, we obtain a formula $\downarrow i.\mathsf{E}\downarrow j.\chi \in \mathcal{L}[\sigma]$ that is easily shown to be $[\vec{p}/\vec{\psi}]$-equivalent to $\phi$ on $\mathsf{F}$ ⊣

PROOF OF THEOREM 4.1.3 . To simplify the induction, we will temporarily extend the syntax of the modal language $\mathcal{L}^1$, by allowing unary predicates to occur as arguments of other predicates. For instance, $R_\triangle(y, P)$ is allowed as a formula, and it is interpreted as $\exists x(Px \wedge R_\triangle yx)$. This change clearly does not affect the expressive power of $\mathcal{L}^1$, but it will make the inductive argument simpler. Besides this, the proof proceeds similarly to that of Theorem 4.1.2 (note that the universal modality is definable in terms of the difference operator: $\mathsf{E}\phi$ is equivalent to $\phi \vee \mathsf{D}\phi$). We will only provide the inductive argument for formulas of the form $\exists y.\psi$.

Let $\phi \in \mathcal{L}^1[\sigma]$ be of the form $\exists y.\psi$. By the definition of $\mathcal{L}^1$, $\phi$ contains at most one free variable, say $x$ (in case $\phi$ contains no free variables, let $x$ be any variable distinct from $y$). Let $q, r$ be distinct propositional variables not in $\sigma$ and distinct from $p_1, \ldots, p_n$. By induction hypothesis, $\phi[x/q, y/r] \in \mathcal{L}^1[\sigma \cup \{p, q\}]$ is $[\vec{p}/\vec{\psi}]$-equivalent on $\mathsf{F}$ to some $\chi \in \mathcal{L}[\sigma \cup \{p, q\}]$. By Lemma 4.3 and the fact that $\mathcal{M}(\mathsf{D}) \subseteq_{\mathsf{F}} \mathcal{L}$, we can obtain a formula $\downarrow p.\mathsf{E}\downarrow q.\chi \in \mathcal{L}[\sigma]$ that is easily shown to be $[\vec{p}/\vec{\psi}]$-equivalent to $\phi$ on $\mathsf{F}$. ⊣

## §5. Applications.

**5.1. Interpolation for $\mathbf{K}_{\mathcal{M}(\mathsf{D})}$ and $\mathbf{K}_{\mathcal{H}(\mathsf{E})}$.** As a corollary of Theorems 4.1.2 and 4.1.3, we obtain the following negative interpolation result for $\mathcal{M}(\mathsf{D})$ and $\mathcal{H}(\mathsf{E})$.

PROPOSITION 5.1. *Let $\mathsf{F}$ be any non-empty frame class closed under disjoint union. Then $\mathcal{M}(\mathsf{D})$ and $\mathcal{H}(\mathsf{E})$ lack interpolation on $\mathsf{F}$.*

PROOF. By Theorems 4.1.2 and 4.1.3, it suffices to show that $\mathcal{M}(\mathsf{D})$ and $\mathcal{H}(\mathsf{E})$ lack full first-order expressivity on $\mathsf{F}$. In fact, we will show that $\exists xyz(Px \wedge Py \wedge Pz \wedge x \neq y \wedge x \neq z \wedge y \neq z)$ is not expressible. Take any frame $\mathfrak{F} \in \mathsf{F}$, and let $w \in \mathfrak{F}$. Let $\mathfrak{G}$ be the disjoint union of five copies of $\mathfrak{F}$, and let $w_1, \ldots, w_5$ be the five copies of $w$ in $\mathfrak{G}$. As $\mathsf{F}$ is closed under disjoint union, $\mathfrak{G} \in \mathsf{F}$. Consider the signature that consists of precisely one propositional variable $p$ and no nominals, and let $V, V'$ be the valuations that send $p$ to $\{w_1, w_2\}$ and $\{w_1, w_2, w_3\}$ respectively. By means of a simple bisimulation argument, one can show that $(\mathfrak{G}, V, w_1)$ and $(\mathfrak{G}, V', w_1)$ are indistinguishable for $\mathcal{H}(\mathsf{E})$ and $\mathcal{M}(\mathsf{D})$. They are however distinguished by $\exists xyz(Px \wedge Py \wedge Pz \wedge x \neq y \wedge x \neq z \wedge y \neq z)$. ⊣

In particular, it follows that interpolation fails for $\mathcal{H}(\mathsf{E})$ and $\mathcal{M}(\mathsf{D})$ relative to any non-empty modally definable frame class.

**5.2. There is no decidable hybrid language with interpolation.** Areces, Blackburn, and Marx [2] pose the question whether there is any decidable hybrid language with interpolation extending $\mathcal{H}(@)$. Since $\mathcal{H}(@, \downarrow)$ is known to be undecidable [2], Theorem 4.1.1 suggests a negative answer. Indeed, there is no decidable hybrid language with interpolation, provided we make one extra natural requirement on the language in question.

DEFINITION 5.1. *A hybrid language $(\mathcal{L}, \models_{\mathcal{L}})$ is a* concrete extension *of $\mathcal{H}(@)$ if the following conditions hold.*

1. *For all signatures $\sigma = (\textsc{prop}, \textsc{nom})$, $\mathcal{L}[\sigma]$ is a set of finite strings over some countable alphabet that includes $\textsc{prop} \cup \textsc{nom} \cup \{\wedge, \neg, \Diamond, \top, @, (, )\}$. Furthermore, $\top \in \mathcal{L}[\sigma]$, $\textsc{prop}, \textsc{nom} \subseteq \mathcal{L}[\sigma]$, and for all $i \in \textsc{nom}$, $\triangle \in \textsc{mod}$ and $\phi, \psi, \phi_1, \ldots, \phi_{n(\triangle)} \in \mathcal{L}[\sigma]$, it holds that $(\phi \wedge \psi), \neg\phi, \triangle(\phi_1, \ldots, \phi_{n(\triangle)}), @_i\phi \in \mathcal{L}[\sigma]$*

2. $\models_{\mathcal{L}}$ *gives the usual interpretation to the propositional variables, nominals, boolean connectives, the diamonds and the @-operators. I.e.,* $\mathfrak{M}, w \models_{\mathcal{L}} \neg\phi$ *iff* $\mathfrak{M}, w \not\models_{\mathcal{L}} \phi$*, etc.*

The following proof is a slight generalization of the undecidability argument for $\mathcal{H}(@, \downarrow)$ given by Areces, Blackburn, and Marx [2].

THEOREM 5.1. *Assume we have at least one unary modality. Every concrete extension of* $\mathcal{H}(@)$ *that has interpolation on the class of all frames has an undecidable satisfiability problem on the class of all frames.*

PROOF. The proof proceeds by reduction of an undecidable problem. For any frame class $\mathsf{F}$ and modal formula $\phi$, we say that $\phi$ is globally satisfiable on $\mathsf{F}$ if there is a model $\mathfrak{M}$ based on a frame in $\mathsf{F}$ such that for all world $w$ of $\mathfrak{M}$, $\mathfrak{M}, w \models \phi$. Spaan [13] proved that global satisfiability of modal formulas on $\mathsf{F}_{23}$, i.e., the class of frames in which every point has at most two successors and at most three two-step successors, is undecidable. We will show that this problem can be reduced to the satisfiability problem for our hybrid language $\mathcal{L}$.

> SPY-POINT LEMMA ( Areces, Blackburn, and Marx [2]). Let $\Sigma$ be any set of first-order frame conditions and let $\phi$ be any modal formula. Then $\phi$ is globally satisfiable on the class of frames satisfying $\Sigma$ iff $\{\textsc{spy}, \Box(\phi^{\neg i}), \psi^* \mid \psi \in \Sigma\}$ is satisfiable, where, for some fixed nominal $i$,

$$
\begin{array}{lcllcl}
\top^{\neg i} & = & \top & (\top)^* & = & \top \\
p^{\neg i} & = & p & (Rxy)^* & = & \Diamond(x \wedge \Diamond y) \\
x^{\neg i} & = & x & (x = y)^* & = & \Diamond(x \wedge y) \\
(\neg\phi)^{\neg i} & = & \neg(\phi^{\neg i}) & (\neg\phi)^* & = & \neg(\phi^*) \\
(\phi \wedge \psi)^{\neg i} & = & \phi^{\neg i} \wedge \psi^{\neg i} & (\phi \wedge \psi)^* & = & \phi^* \wedge \psi^* \\
(\Diamond\phi)^{\neg i} & = & \Diamond(\neg i \wedge \phi^{\neg i}) & (\exists x.\phi)^* & = & \Diamond{\downarrow}x.\Diamond(i \wedge \phi^*)
\end{array}
$$

$$\textsc{spy} = i \wedge \Box\neg i \wedge \Box\Diamond i \wedge \Box\Box{\downarrow}y(\neg i \rightarrow \Diamond(i \wedge \Diamond y))$$

Clearly, the class of frames $\mathsf{F}_{23}$ can be defined by a first-order formula $\psi_{23}$. By the Spy-Point Lemma, it follows that for all modal formulas $\phi \in \mathcal{L}[\sigma]$, $\phi$ is globally satisfiable on $\mathsf{F}_{23}$ iff $\textsc{spy} \wedge (\psi_{23})^* \wedge \Box(\phi^{\neg i})$ is satisfiable.

By Lemma 4.2, $\textsc{spy} \wedge (\psi_{23})^*$ it is equivalent to some formula $\gamma \in \mathcal{L}[\sigma \cup \{i\}]$. Since $\mathcal{L}$ is a concrete extensions of $\mathcal{H}(@)$, $\Box(\phi^{\neg i}) \in \mathcal{L}[\sigma \cup \{i\}]$ and $\mathcal{L}[\sigma \cup \{i\}]$ is closed under conjunction. The translation from $\phi$ to $\Box(\phi^{\neg i})$ is effective (in fact linear), and the formula $\gamma$ is constant: it doesn't depend on $\phi$. Therefore, we have effectively reduced the global satisfiability problem for modal formulas on $\mathsf{F}_{23}$ to the satisfiability problem of $\mathcal{L}$. It follows that satisfiability for $\mathcal{L}$ is undecidable. ⊣

**5.3. Arrow logic and relation algebra.** In this section, we consider the basic modal language over a collection of three modalities: a binary modality $\circ$, a unary modality $\otimes$, and a null-ary modality (modal constant) $\delta$. Thus, the formulas of this language are given by

$$\phi ::= p \mid \top \mid \neg\phi \mid \phi \wedge \psi \mid \phi \circ \psi \mid \otimes\phi \mid \delta$$

The corresponding frames have three accessibility relations, one for each modality. Let $\mathsf{SQ}$ be the class of such frames $\mathfrak{F} = (W, R_\circ, R_\otimes, R_\delta)$ for which there is a set $U$ such that $W = U \times U$, and

$$\begin{array}{lcl} R_\circ & = & \{((w,v),(w,u),(u,v)) \mid w,v,u \in U\} \text{ (i.e., } R_\circ \text{ denotes composition)} \\ R_\otimes & = & \{((w,v),(v,w)) \mid w,v \in U\} \text{ (i.e., } R_\otimes \text{ denotes inverse)} \\ R_\delta & = & \{(w,w) \mid w \in U\} \text{ (i.e., } R_\delta \text{ denotes the identity relation on } U) \end{array}$$

The basic modal logic of the frame class $\mathsf{SQ}$ is known as *arrow logic*. It is closely related to the class of representable relation algebras. Arrow logic is known not to have interpolation. Using Theorem 4.1.3, we can show something much stronger: the first-order correspondence language is the smallest extension of the basic modal logic that has interpolation on $\mathsf{SQ}$.

Theorem 5.2. *$\mathcal{L}^1$ is the least expressive extension of $\mathcal{BML}$ with interpolation on $\mathsf{SQ}$.*

Proof. The difference operator is definable relative to $\mathsf{SQ}$: for any formula $\phi$, $\mathsf{D}\phi$ is equivalent to $(\neg\delta \circ \phi \circ \top) \vee (\top \circ \phi \circ \neg\delta)$. Hence, $\mathcal{M}(\mathsf{D}) \subseteq_{\mathsf{SQ}} \mathcal{BML}$. The result follows by Theorem 4.1.3. That $\mathcal{L}^1$ has interpolation relative to $\mathsf{SQ}$ follows immediately from the fact that $\mathsf{SQ}$ is an elementary frame class. ⊣

In fact, in order to restore interpolation, Marx [12] has proposed an extension of the language of arrow logic, called $RL{\downarrow}$, which he showed to be equally expressive as $\mathcal{L}^1$ (on $\mathsf{SQ}$). Roughly speaking, Theorem 5.2 tells us that, in terms of expressivity, this is "the cheapest way to restore interpolation".

As was mentioned already, arrow logic is closely related to the class of representable relation algebras. We can rephrase Theorem 5.2 in more algebraic terms by observing that every elementary operation on binary relations is definable in $\mathcal{L}^1$ over $\mathsf{SQ}$. To make this precise, we need to introduce some terminology. Every first-order formula of the form $\phi(R_1, \ldots, R_n, x, y)$, where $R_1, \ldots, R_n$ are binary relation symbols, defines an $n$-ary operation $\mathcal{O}$ on binary relations: for all binary relations $R_1, \ldots, R_n$ on a set $D$, $\mathcal{O}(R_1, \ldots, R_n) = \{(d,e) \in D \mid (D, R_1, \ldots, R_n) \models \phi[d,e]\}$. Operations on binary relations that are defined by a first-order formula in this way are called *elementary*. Examples are intersection $(R_1xy \wedge R_2xy)$, complement $(\neg Rxy)$ and composition $(\exists z.(R_1xz \wedge R_2zy))$.

Proposition 5.2. *Let $\mathcal{O}$ be any $n$-ary elementary operation on binary relations ($n \geq 0$). Then there is a formula $\chi(p_1, \ldots, p_n) \in \mathcal{L}^1[\{p_1, \ldots, p_n\}]$ (involving the modalities $\circ$, $\otimes$ and $\delta$), such that for all models $\mathfrak{M}$ based on a frame in $\mathsf{SQ}$, $[\![\chi(p_1, \ldots, p_n]\!]^{\mathfrak{M}}_{\mathcal{L}^1} = \mathcal{O}([\![p_1]\!]^{\mathfrak{M}}_{\mathcal{L}^1}, \ldots, [\![p_n]\!]^{\mathfrak{M}}_{\mathcal{L}^1})$.*

Proof. Let $\phi(R_1, \ldots, R_n, x, y)$ be any first-order formula defining a map from $n$ binary relations to a single binary relation. Pick corresponding propositional variables (unary predicates) $P_1, \ldots, P_n$, and define $\phi^*$ inductively as follows

$$\begin{array}{lcl} (R_kxy)^* & = & \exists z.(P_k(z) \wedge R_\circ zxz \wedge R_\circ zzy) \\ (x = y)^* & = & x = y \\ \top^* & = & \top \\ (\phi \wedge \psi)^* & = & \phi^* \wedge \psi^* \\ (\neg\phi)^* & = & \neg(\phi^*) \\ (\exists x.\phi)^* & = & \exists x.(R_\delta(x) \wedge \phi^*) \end{array}$$

Finally, let $\chi(x) \in \mathcal{L}^1[\sigma]$ be the formula $\exists yz.(\phi^*(y,z) \wedge R_\circ xyx \wedge R_\circ xxz)$. Then for all models $\mathfrak{M}$ based on a frame in $\mathsf{SQ}$, $[\![\chi(p_1,\dots,p_n)]\!]^{\mathfrak{M}}_{\mathcal{L}^1} = \mathcal{O}([\![p_1]\!]^{\mathfrak{M}}_{\mathcal{L}^1},\dots,[\![p_n]\!]^{\mathfrak{M}}_{\mathcal{L}^1})$. The proof of this claim is left to the reader. ⊣

We can conclude from this that, algebraically speaking, the only way to restore interpolation for the class of representable relation algebra by expansion is to add the entire clone of elementary operations on binary relations. In particular, it does not suffice to add only finitely many elementary operations, or to add only Jónsson's Q-operators [15]. Similar results can be obtained for cylindric algebras of finite dimension. Again, the conclusion is that first-order logic is the least expressive extension with interpolation.

**5.4. Guarded fragment.** So far, the main results were stated in terms of abstract modal languages. However, there is another perspective in terms of *fragments of first-order logic*. In particular, Theorem 4.1.2 can be reformulated as follows (where FREE($\phi$) denotes the set of variables occurring freely in the first-order formula $\phi$):

COROLLARY 5.1. *Consider first-order languages with equality and constants but without function symbols of arity $\geq 2$. Let $F$ be any fragment of first-order logic satisfying the following conditions.*

1. *all atomic formulas (i.e., formulas of the form $R\vec{t}$ or $t_1 = t_2$) are in $F$.*
2. *if $\phi, \psi \in F$ then $\neg\phi \in F$ and $\phi \wedge \psi \in F$.*
3. *if $\phi$ is an atomic formula, $\psi \in F$, FREE($\psi$) $\subseteq$ FREE($\phi$) and $\vec{x}$ is a sequence of variables, then $\exists\vec{x}(\phi \wedge \psi) \in F$*
4. *$F$ has interpolation: for all $\phi, \psi \in F$ with at most one free variable, if $\models \phi \to \psi$ then there is a $\theta \in F$ such that $\models \phi \to \theta$, $\models \theta \to \psi$ and all non-logical symbols occurring in $\theta$ occur both in $\phi$ and in $\psi$.*

*Then every first-order formula with at most one free variable is equivalent to a formula of $F$.*

PROOF. Any fragment $F$ satisfying the above requirements constitutes a modal language in the following sense. For any signature $\sigma = (\text{PROP}, \text{NOM})$, let $\sigma^*$ be the first-order signature that has PROP as its unary predicates, NOM as its constants, and that has a relation $R_\triangle$ of arity $n(\triangle)+1$ for each $\triangle \in$ MOD (here we assume again a fixed, given set of modalities MOD. Fix a first-order variables $x$, and for all signatures $\sigma$, let $\mathcal{L}_F[\sigma]$ be the collection of first-order formulas in the signature $\sigma^*$ that are in the fragment $F$. Furthermore, let $\mathfrak{M}, w \models_{\mathcal{L}_F} \phi(x)$ iff $\phi(x)$ holds in $\mathfrak{M}$ conceived of as a first-order structure, interpreting $x$ as $w$ and $R_\triangle$ as the accessibility relation for $\triangle$. Then $(\mathcal{L}_F, \models_{\mathcal{L}_F})$ is a modal language.

From the requirements given above, it follows in fact that $\mathcal{L}_F$ extends $H(\mathsf{E})$ (the proof is straightforward, by induction), and that $\mathcal{L}_F$ has interpolation on the class of all frames. Consequently, Theorem 4.1.2 applies and we can conclude that $\mathcal{L}^1 \subseteq \mathcal{L}_F$. In other words, every first-order formula with at most one free variable $x$ is equivalent to a formula in the fragment $F$. ⊣

This result applies to the guarded fragment with constants, as defined by Grädel [10]. It also applies to the packed fragment, the loosely guarded fragment and the clique-guarded fragment, provided that constants are added to these languages. Note that Corollary 5.1 is a weakening of Theorem 4.1.2, since it only applies to fragments of first-order logic, thus excluding for instance second-order or infinitary languages.

Without proof, we state two straightforward generalization of this result. Firstly, Hoogland and Marx [11] show that, while interpolation fails for the Grädel-style guarded fragment, the *purely relational* guarded fragment (i.e., without constants) does satisfy a weak version of interpolation that is strong enough to entail the Beth property. Corollary 5.1 can be shown to apply also to this weak version of interpolation, provided that constants are allowed again.

Secondly, in the original definition of the guarded fragment by Andréka, Benthem, and Németi [1], identity statements are not allowed as guards (i.e., all quantifiers must be guarded by atomic formulas of the form $Rt_1 \dots t_n$). Assuming that constants are allowed, the least expressive extension of this version of the guarded fragment with interpolation is precisely what Andréka, Benthem, and Németi [1] refer to as the fragment $F3$.

**§6. Discussion.** Various modal and hybrid languages are characterized in terms of interpolation. We show that very few languages involving nominals or the difference operator have interpolation. As a corollary, we show that no decidable hybrid language has interpolation on the class of all frames, and that the only way repair interpolation for arrow logic or for the guarded fragment with constants it to extend them to full first-order logic.

The proofs make essential use of tools from hybrid logic (in particular, the ↓-binder). This demonstrates that hybrid logic can make contributions to the wider study of modal logic and fragments of first-order logic.

Incidentally, it has recently been shown [14] that $\mathcal{H}(@)$ has *interpolation over propositional variables*, and hence the Beth property, on the class of all frames (and indeed on many other frame classes). According to this weak notion of interpolation, nominals are allowed to occur freely in the interpolant, like modalities.

We finish with an open problem. An interesting hybrid language that we did not discuss in this paper is the extension of the basic modal language with only nominals, no @-operators. We could call this the *minimal hybrid language* $\mathcal{H}$.

QUESTION 6.1. *What is the least expressive modal language with interpolation extending* $\mathcal{H}$*?*

We have some partial results in this direction: one can show that the language in question is at least as expressive as iteration-free PDL with intersection plus nominals, graded modalities and the ↓-operator. As an upperbound, we know that the language is at most as expressive as $\mathcal{H}(@, \downarrow)$ (in fact, one can show that $\mathcal{H}(@, \downarrow)$ is the smallest extension of $\mathcal{H}$ having a strong version of interpolation according to which also the modalities occurring in the interpolant must be in the common language). Finally, one can also show that the language in question has an undecidable satisfiability problem, by analogy to the Theorem 5.1.

REFERENCES

[1] Hajnal Andréka, Johan van Benthem, and István Németi, *Modal logics and bounded fragments of predicate logic*, ***Journal of Philosophical Logic***, vol. 27 (1998), no. 3, pp. 217–274.

[2] Carlos Areces, Patrick Blackburn, and Maarten Marx, *Hybrid logics: Characterization, interpolation and complexity*, ***The Journal of Symbolic Logic***, vol. 66 (2001), no. 3, pp. 977–1010.

[3] Jon Barwise and Solomon Feferman, ***Model-theoretic logics***, Springer, New York, 1985.

[4] Patrick Blackburn, Maarten de Rijke, and Yde Venema, ***Modal logic***, Cambridge University Press, Cambridge, UK, 2001.

[5] Willem Conradie, *Definability and changing perspectives: The Beth property for three extensions of modal logic*, ***Master's thesis***, ILLC, University of Amsterdam, 2002.

[6] William Craig, *Linear reasoning. a new form of the Herbrand-Gentzen theorem*, ***The Journal of Symbolic Logic***, vol. 22 (1957), pp. 250–268.

[7] Maarten de Rijke, *The modal logic of inequality*, ***The Journal of Symbolic Logic***, vol. 57 (1992), no. 2, pp. 566–584.

[8] Dov Gabbay, *Craig's interpolation theorem for modal logics*, ***Conference in mathematical logic — London '70*** (Wilfrid Hodges, editor), Lecture Notes in Mathematics, Springer-Verlag, 1972, pp. 111–127.

[9] George Gargov and Valentin Goranko, *Modal logic with names*, ***Journal of Philosophical Logic***, vol. 22 (1993), no. 6, pp. 607–636.

[10] Erich Grädel, *On the restraining power of guards*, ***The Journal of Symbolic Logic***, vol. 64 (1999), no. 4, pp. 1719–1742.

[11] Eva Hoogland and Maarten Marx, *Interpolation and definability in guarded fragments*, ***Studia Logica***, vol. 70 (2002), no. 3, pp. 373–409.

[12] Maarten Marx, *Relation algebra with binders*, ***Journal of Logic and Computation***, vol. 11 (2001), no. 5, pp. 691–700.

[13] Edith Spaan, *Complexity of modal logics*, ***Ph.D. thesis***, ILLC, University of Amsterdam, 1993.

[14] Balder ten Cate, Maarten Marx, and Petrucio Viana, *Sahlqvist theory for hybrid logic*, ***Liber amicorum Dick de Jongh*** (Anne Troelstra, editor), University of Amsterdam, 2004.

[15] Yde Venema, *Relational games*, ***Algebraic logic*** (Hajnal Andréka, Donald Monk, and István Németi, editors), 1991.

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