Fine Grained Theories of Time

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In recent years there has been a fruitful exchange of ideas between philosophical logic and Artificial Intelligence (AI), most notably in the study of logics of belief. In the case of temporal logic, however, interaction has been minimal. This is surprising. Central areas of AI (for example, planning) raise interesting issues about temporal representation and reasoning, and the philosophical tradition (most notably the modal approach to temporal logic pioneered by Arthur Prior [40]) has provided detailed maps of temporal logic. But, by and large, philosophical logic and AI have gone separate ways.

Why is this? An examination of some well-known AI temporal representation formalisms, such as those of Allen [1] and McDermott [33], suggests some answers. For a start, both Allen and McDermott take as fundamental the ability to *refer* to times: one names a time and asserts that something holds, or does not hold, *at that particular time* — but temporal reference isn't possible in Prior's original tense logic, nor in the richer interval-based tense logics subsequently developed. Furthermore, the AI tradition takes the *sorted* nature of temporal information seriously. For example, a central motivation for Allen's approach is to capture the distinction between properties, events, and processes. But, while sortal distinctions play an important role in natural language semantics (see, for example, Vendler [53] and Dowty [21]), in the logical tradition they have been rather marginal. Sorting a first-order language does not lead to new expressivity,[1] so most logicians have paid little attention to sortal diversity.

The main claim of the present paper is that the divergence of interests between philosophical logic and AI is only apparent. By sorting the atomic symbols of various modal logics of time we will obtain systems in which reference to times is possible, and in which the ontological variety important in both AI and natural language semantics can be mirrored. In an extended case study we'll examine Allen's temporal logic, first through the eyes of sorted Priorean tense logic, and then via a sorted version of Halpern and Shoham's [32] interval-based logic. We'll succeed in capturing all the main ideas of Allen's system, and will do so in a way that avoids the problematic aspects of his account of properties. Moreover, as we shall see, sorting *modal* languages does not yield syntactic sugar: it brings about genuine expressive gains, and these gains are relevant to

[1] Any formula of a sorted first-order language can be regarded as an abbreviation of an ordinary first-order formula over a richer signature; see Enderton [22].

temporal reasoning.

But the paper has a secondary aim: to explore the idea that natural languages can guide the design of knowledge representation formalisms. The fundamental difficulty of applying logic to AI is not so much finding *some* logical representation, but finding a *good* one. The logical landscape is vast: how do we locate the places of interest? The idea that natural languages might be a useful guide is appealing (see, for example, the remarks in Chapter 6 of Turner [51]) and this paper tries to takes it seriously. The sorted systems considered here were not developed for use in AI; they are parts of richer languages designed with the needs of natural language semantics in mind (see Blackburn [5, 8]). Only subsequently was it observed that these systems offered an interesting perspective on temporal knowledge representation.

We proceed as follows. We first review Priorean tense logic, emphasizing its natural language origins, and noting the (surprising) blend of expressive strength and weakness it offers. In the second section we address its principal weaknesses (the lack of temporal reference), and show how to overcome it via sorting; the upshot is a simple extension of Priorean tense logic in which reference to both points and intervals is possible. In the third section we make our first attempt to model Allen's framework. The attempt is partially successful: we are able to model Allen's account of properties. However, although our sorted tense logic can refer to intervals, its semantics remains point-based, and this blocks a sortal account of Allen's treatment of events. So, in the fourth section we take the obvious step: we apply the sorting strategy to a richer interval-based tense logic, the logic of Halpern and Shoham [32].

The sorted modal languages used in this paper are a special case of what are now known as *hybrid languages*. Like tense logic itself, hybrid languages, and the idea of sorting on which they build, trace back to the work of Arthur Prior. Work on hybrid languages has blossomed in recent years (a historical sketch is given at the end of Section 2), and the paper concludes with some general remarks on hybridization and temporal knowledge representation.

1 Priorean tense logic

What is the logic of time that underlies everyday discourse? Philosophers have given a variety of answers. According to Quine [42], natural languages contain no special logic of time of any great interest: tenses are simply idiosyncrasies, and should be paraphrased away. Arthur Prior's vision was very different. According to Prior, there are interesting logics of time, and the mechanisms natural languages use to encode temporal information deserve serious study. His views led him to develop *tense logic*, a simple bi-modal language for representing and reasoning about temporal information.

Two main ideas fire Priorean tense logic, and both are abstractions from natural language. The first is syntactic. Prior observed that natural language tenses work rather like modal operators. Tenses 'operate' on sentences, enabling us to situate the state of affairs described in the underlying sentence in various temporal locations (for example, the past, the present, or the future). The second (and deeper) idea is semantic. Prior observed that everyday discourse presupposes an internal, or observer centered, view of events. Past, present and future are not absolute notions: they make sense only relative to a context. Tenses, and temporal indexicals such as today and yesterday, exploit the fact that everyday discourse is temporally situated: by uttering a sentence we implicitly fix a time (the *utterance time*) and natural languages typically specify temporal information relative to this important *deictic centre.*[2] This semantic intuition is neatly mirrored in the Kripke semantics for Priorean tense logic.[3]

Let's make our discussion precise. A language of (propositional) Priorean tense logic is an ordinary language of propositional calculus augmented by two unary modalities F (the *future* tense operator) and P (the *past* tense) operator. That is, Priorean languages contain a denumerably infinite set VAR of distinct propositional variables (typically written p, q, r, and so on), the two punctuation symbols) and (, some truth functionally adequate collection of Boolean connectives (for example $\wedge$ and $\neg$), and F and P.

We make formulas (or sentences) from these symbols as follows. Firstly we stipulate that the set of atomic symbols of the language (hereafter called ATOM) is to be precisely VAR, the set of propositional variables. We then define WFF, the set of well formed formulas of the language to be the smallest set such that: ATOM $\subseteq$ WFF; for all $\phi, \psi \in$ WFF, $(\phi B \psi)$ and $\neg\phi \in$ WFF, where B is any binary Boolean operator; and for all $\phi \in$ WFF, $F\phi$ and $P\phi \in$ WFF. A wff of the form $F\phi$ is read as 'it will be the case that ϕ' (that is, ϕ will hold in the future) and a wff of the form $P\phi$ as 'it was the case that ϕ' (that is, ϕ did hold in the past). We introduce two defined operators G and H, the duals of F and P respectively. $G\phi$ is defined to be $\neg F \neg\phi$ and $H\phi$ is defined to be $\neg P \neg \phi$. $G\phi$ reads 'it is always *going* to be the case that ϕ', and $H\phi$ reads 'it always *has*

[2] For a linguistic analysis of tense based on the notion of deictic centre, see Comrie [18].

[3] This brief sketch does little justice to Prior's conception. For a detailed (and elegant exposition, Prior's book *Past, Present and Future* [40] is particularly good. Incidentally, it's not historically anachronistic to treat Kripke semantics as if it was part and parcel of Prior's conception. Not only were the intuitions underlying Kripke semantics shared by Prior (as a reading of *Past, Present and Future* will make abundantly clear , Prior was actually one of the pioneers in its development: as early as 1955 Prior had used what we would now call Kripke models over the frame $\langle N, < \rangle$ (the natural numbers in their usual order ; see pages 22–23 of *Past, Present and Future* for further details.

One word of warning. Prior writes elegantly and clearly, but if you are unused to philosophical argumentation, you may him find him tough going. But Prior can't be understood by relying purely on secondary sources: relatively few commentators on his work in temporal logic seem to have read him in detail, and sometimes (particularly in discussions of natural language semantics notions are ascribed to Prior which bear little or no resemblance to his views. One secondary source worth noting is Ohrstrom and Hasle [35]; indeed this would be an excellent adjunct to the present paper. Nonetheless, Prior was a highly original thinker, arguably one of the key philosophical logicians of the 20th century, and there is no substitute for tackling his (often demanding writings in the original.

been the case that ϕ'. The *length* of a formula ϕ (written $|\phi|$) is the number of primitive symbols it contains.

The semantics of Priorean tense logic is defined in terms of *frames* and *Kripke models*. A frame $\mathbf{T}$ is an ordered pair $\langle T, < \rangle$ where T is a non-empty set, and $<$ is a binary relation on T. The elements of T are thought of as instants (or points) of time and $<$ is thought of as the relation of temporal precedence: if $t < t'$ we say that t' is later that t, or t is earlier than t'. In short, frames are a simple set theoretic rendition of a pervasive view of time: that time is rather like a river, a flow of temporal elements from earlier to later. Actually, because we have placed no constraints on the temporal ordering $<$, we're admitting 'flows' that look more like swamps than rivers. No matter. We'll define our semantics in terms of arbitrary frames, but when it comes to applications we shall restrict ourselves to working with a *linear* conception of time. To be more precise, we shall then assume that we are dealing only with frames $\langle T, < \rangle$ that are *strict total orders* (STOs).[4]

Given a frame $\mathbf{T}$ $(= \langle T, < \rangle)$, a *valuation* V on $\mathbf{T}$ is a function $V : ATOM \longrightarrow Pow(T)$. That is, a valuation assigns to each atom a set of times, the times where that particular piece of atomic information is true. In short, *valuations specify* mantics is built on internal, observer centered, ideas: the fundamental semantic notion, $\mathbf{M}, t \models \phi$, relativizes truth to a time t, the *point of evaluation*, and we can think of this relativization as a first attempt to capture the notions of utterance time and deictic center. Furthermore, the satisfaction definition treats the tense operators as devices which scan the states $<$-accessible from the point of evaluation for information. This models one of the most important facts about tensed sentences, their context dependence.

Now that we know what Priorean tense logic is, and something of the intuitions that guided its design, a different type of question beckons: what exactly is Prior's formalism when viewed from a *logical* perspective? The answer, a rather surprising one, emerged in the 1970s: Priorean tense logic is essentially a fragment of second-order logic.[5]

The second-order nature of Priorean tense logic reveals itself when the following question is raised: what general constraints on the flow of time can be imposed using Priorean wffs? Put more simply, what can the Priorean language say about *frames*? Now, it's straightforward to see that certain first-order constraints are expressible. For example, its easy to see that $FFp \rightarrow Fp$ is valid on precisely the frames that are transitive. That is, for any frame $\mathbf{T}$ $(= \langle T, < \rangle)$, we have that $\mathbf{T} \models FFp \rightarrow Fp$ iff $<$ is transitive, and thus we say that $FFp \rightarrow Fp$ *defines* transitivity. As a second example, consider *density*.[6] This is a natural constraint to impose for many applications, and it is definable by means of a Priorean wff: $\mathbf{T} \models Fp \rightarrow FFp$ iff $<$ is dense.

On the other hand some very simple first-order constraints on the flow of time are *not* definable using Priorean wffs: in fact neither *irreflexivity*, *antisymmetry*, *asymmetry*, *right directedness*, *trichotomy*, nor *right discreteness* are definable.[7] These constraints are relevant to temporal reasoning: irreflexivity, antisymmetry and asymmetry forbid various kinds of temporal whirlpools, right directedness points time's river towards the future, trichotomy underlies the linear conception of time, and discrete time models may be the models that best reflect physical reality. Thus, regarded as a medium for talking about frames, the Priorean language has some striking weaknesses.

Now for the surprise. Although Priorean languages have these first-order limitations, they nonetheless succeed in stepping across the boundary into second-

[5] The detailed development of this answer is due to several logicians, including Thomason [48, 49, 50], Fine [23], Goldblatt [30] and van Benthem [3]; this work is still some of the most interesting in modal logic. The first frame incompleteness result (which signaled the presence of second-order phenomena was proved (by Thomason [48] in the setting of Priorean tense logic, but a second-order perspective is needed for most modal languages. For further discussion and history, see Blackburn, de Rijke and Venema [10].

[6] That is, $\forall t \forall t'(t < t' \rightarrow \exists s(t < s \wedge s < t'$.

[7] Irreflexivity and trichotomy were defined in Footnote 3. Antisymmetry means $\forall t \forall t'(t < t' \wedge t' < t \rightarrow t = t'$, asymmetry means $\forall t \forall t'(t < t' \rightarrow t' \not< t$, right directedness means $\forall t \forall t' \exists s(t < s \wedge t' < s$, and right discreteness means $\forall t \forall t'(t < t' \rightarrow \exists s(t < s \wedge \neg \exists s'(t < s' < s$. I won't prove that these properties are not definable (this would take up too much space but the examples are all standard, and for further discussion and proofs the reader is advised to consult van Benthem [4], or Blackburn, de Rijke and Venema [10].

order logic: they can express constraints that no first-order language can. Van Benthem [4] gives a number of nice examples. One of them is ϕ^Z, a conjunction of five Priorean formulas, one of which is the following rather mysterious looking wff (actually a bidirectional variant of the Löb formula used in modal provability logic):[8]

$$(H(Hp \to p) \to (PHp \to Hp)) \wedge (G(Gp \to p) \to (FGp \to Hp)).$$

As van Benthem shows, ϕ^Z comes close to defining $\mathbf{Z}$ $(= \langle Z, < \rangle)$, the set of integers in their usual order, up to isomorphism. To be more precise, he shows that ϕ^Z defines $\mathbf{Z}$ up to isomorphism on the class of strict total orders: that is, ϕ^Z is valid on a frame $\mathbf{T}$ that is a STO iff $\mathbf{T}$ is isomorphic to $\mathbf{Z}$. This is something that cannot be done in first-order logic.[9] The magic lies in the variant of Löb's formula. On transitive frames this guarantees that only a *finite* number of points lie between any pair of points t and t' of the frame. This is a second-order constraint, and it is its ability to enforce this property that enables Priorean tense logic to come so close to defining $\mathbf{Z}$.

For further discussion of tense logic as second-order logic (and in particular, for an account of *why* this perspective exists at all) the reader should consult van Benthem [4]. Here I merely wish to draw a methodological moral. Prior was lead to tense logic for philosophical reasons: he believed that only a logic based on internal, observer centered, ideas could do justice to our conception of time and the language we use to talk about it. This lead him to reject classical analyses and explore the use of modal languages. But the consequences of this (seemingly modest) decision are dramatic: we wind up with a temporal language that cuts the cake of expressivity on strikingly new lines. Evidently natural language can lead us in unexpected directions, so the question to pose next is: what else does it have to teach us?

2 Referential tense logic

We don't have to look far for further lessons. Although Priorean tense logic neatly captures the deictically centered nature of natural language tenses, it fails to get to grip with a linguistic fact of equal importance: tenses are typically *referential*. For example, an utterance of Vincent accidentally squeezed

[8]The other conjuncts of ϕ^Z are $FFp \to Fp$ (which, as has already been mentioned, defines transitivity , $P\top \wedge F\top$ (which defines those frames in which there is no first point of time and no last point of time; $\top$ here is shorthand for $p \vee \neg p$, $F(p \wedge q \to F(p \wedge Fq \vee F(p \wedge q \vee F(q \wedge Fp$ (this defines the class of frames in which time doesn't branch to the future , and $P(p \wedge q \to P(p \wedge Pq \vee P(p \wedge q \vee P(q \wedge Pp$ (this defines the class of frames in which time does not branch towards the past .

[9]Any first-order formula φ (or indeed, any set Σ of first-order formulas which putatively has the same effect as ϕ^Z has at least one infinite model, namely $\mathbf{Z}$ itself. Thus by the upward Löwenheim Skolem theorem, φ (or Σ must have models of every infinite cardinality. The uncountable models cannot be isomorphic to $\mathbf{Z}$.

the trigger doesn't mean that at some completely unspecified past time Vincent did in fact accidentally squeeze the trigger, it means that at some *particular*, contextually determined, past time he did so.[10] But the referential aspect of the English simple past tense cannot be straightforwardly mirrored in Priorean languages. The natural representation we have for the trigger-squeezing sentence is P(Vincent accidentally squeeze the trigger), but while this correctly places the accidental squeezing in the past, the referential force of the English original eludes it. Although Prior abstracted tense logic from natural language, nowadays formal semanticists regard his abstraction as rather crude, and prefer to work with systems such as Kamp's DRT in which temporal reference is possible.

So let's extend Priorean tense logic to make temporal reference possible. Now, in extending tense logic we have to be careful not to destroy those features that made it interesting in the first place. Priorean tense logic is a *simple* system motivated by general features of natural language. We should look for extensions that involve minimal tampering with the syntax and semantics of the Priorean original, and be guided by natural language in our search. An idea that emerges naturally from these desiderata is to *sort* Priorean tense logic.

Here's an important intuition about natural language: different sorts of information comes in different sorts of packets, and knowing what kinds of packets there are, and what types of information they contain, is an important part of knowing a language. To give a simple example, suppose that while shopping in a crowded department store you overhear a snippet of conversation: '...runs on Sunday...'. Now, you wouldn't be able to tell anybody *what* was running: you didn't hear that part of the conversation, and the topic might have been a person, a train, a course, or an egg. Nor could you say *when* the running took place or will take place: certainly on a Sunday, but which one? Indeed no particular Sunday may have been referred to: if the habitual reading of the present tense was intended, the intent may have been to quantify across several Sundays. Nonetheless, as even this discussion of what we don't know shows, we certainly can glean *something* from the overheard words: minimally that some process (a running) is supposed to take place (either regularly or once) on the day of the week that is neither Monday, Tuesday, ..., nor Saturday. Languages are highly structured, and even on the basis of rather minimal input, our knowledge of this structure may enable us to deduce part of what was (or was not) intended. Part of this structure involves semantic *classifications* and *contrasts*. For example we know that 'Sunday' can be classified as a day-naming-word, and that its denotation contrasts with the denotation of other day-naming-words such as 'Tuesday'. Knowing the various classifications and contrasts and how they are encoded is an important part of knowledge of language, and it is this (rather

[10]For example, if the previous sentence had been The car jolted unexpectedly, we would normally interpret Vincent accidentally squeezed the trigger to mean that Vincent accidentally squeezed the trigger at, or just after, the time of the unexpected jolt. For a good introduction to tense, temporal reference, and their interaction, couched in terms of Kamp's Discourse Representation Theory (DRT , see Kamp and Reyle [36].

Saussurean) intuition that I'm going to cartoon in tense logic.

Sorted first-order languages are well understood, but what on earth is a sorted *modal* language? Let's consider the matter. Like most modal languages, Priorean tense logic lets us build information representations using Boolean operators and modalities. Ultimately, complex representations are built up out of the atomic symbols. However in Priorean languages, as in other modal languages, there is only one kind of atomic symbol: propositional variables. Why not introduce a second sort of atomic sentence, syntactically distinct from the first? Indeed, why not introduce a third, a fourth or a fifth new sort of atomic sentence, each distinct from the propositional variables and each other? Syntactically these items would be on a par with the familiar propositional variables: we'd build up sentences using these new symbols, or some mixture of these new symbols and propositional variables, exactly as before. Semantically, however, each sort would be distinct: each sort would encode a different type of information, or to put it another way, each sort would have a different characteristic intended meaning. These characteristic meanings would be captured by placing constraints on the functions that were permitted to interpret the items in each sort: only those functions which respected the intended meaning would qualify as valuations.

Let's apply this idea to the problem at hand. We'll first add a new sort to Priorean tense logic to make it 'point referential'. So, let NOM, the set of *nominals*, be a set of symbols such that VAR$\cap$NOM$= \emptyset$. We typically represent nominals by the letters i, j, k and so on. Define the set of atoms of our language, ATOM, to be VAR$\cup$NOM. This single atomic level change is the *only* syntactic change we make: the wffs are made up out of the elements of ATOM exactly as for Priorean languages. That is, all atoms are wffs, and we can freely form new wffs from other wffs using the Boolean connectives and tense operators. The only difference is that now the atomic symbols are subdivided into two categories, nominals and propositional variables. Note that the wffs fall naturally into three classes. First there are wffs like $FFp \to Fp$, which contain only propositional variables. We'll call these *purely Priorean* wffs. Second there are wffs like $i \to \neg Fi$ which contain only nominals. We'll call these *purely nominal* wffs. Finally there are wffs such as $F(i \wedge p)$ which contain both sorts of atom, and we'll call these *mixed* wffs.

We want nominals to enable us to refer to points of time. To achieve this we're going to insist that in any model, *nominals are to be true at one and only one time*. Nominals will 'name' the unique time they are true at. So, let $\mathbf{T}$ be any frame. By a valuation for our language is meant a function $V : ATOM \longrightarrow Pow(\mathbf{T})$ such that for all $i \in$ NOM, $V(i)$ is a singleton. This atomic level change is the *only* semantic change we make. As with Priorean tense logic we define models $\mathbf{M}$ to be pairs $\langle \mathbf{T}, V \rangle$ where $\mathbf{T}$ is a frame and V a valuation, and we define the concept of a wff ϕ being satisfied in a model $\mathbf{M}$ at a point t (that is, $\mathbf{M}, t \models \phi$), and the various other semantic notions, *exactly* as we did for Priorean languages.

Thus we have our first sorted modal language (or *hybrid language*, as they are usually called). The language has two sorts, nominals and variables, and is called *nominal tense logic*. Nominals encode information about the identity of times: on encountering a nominal we may not know which time it names, but we *do* know that it names some (unique) time. We inherited propositional variables from tense logic. They encode arbitrary information, and on encountering one we are none the wiser.

What has sorting achieved? In a nutshell: expressivity. For a start, we have complete freedom to use the standard logical to combine referential and other information as we see fit. For example, using the wff

$$P(i \wedge \textsf{Vincent accidentally squeeze the trigger})$$

we can insist not only that Vincent did accidentally squeeze the trigger, but that he did so at a particular time, the time named by i. That is, the *syntactic* encoding of information is completely uniform: everything is a formula. The symmetry is broken only where it really counts: at the *semantic* level.

The expressive increase is reflected in a crop of new validities. For example, consider the purely Priorean wff:

$$F(p \wedge q) \wedge F(p \wedge r) \rightarrow F(q \wedge r).$$

This is *not* valid. From the information that there is a future time where p and q are true, and also a future time where p and r are true, we cannot conclude that there is a future point where q and r are true together. However the following mixed analog *is* valid:

$$F(i \wedge q) \wedge F(i \wedge r) \rightarrow F(q \wedge r).$$

There is only one point at which i is true, so given the truth of the antecedent, the truth of the consequent follows.

But the expressive gains run deeper: with the aid of nominals we can enforce many new constraints on frames. Recall the six examples given earlier of frame properties not definable in Priorean languages. All six conditions are definable in our enriched language (indeed, definable using only purely nominal formulas) as the following result shows:

Theorem 2.1 *Let* $\mathbf{T} = \langle T, < \rangle$ *be a frame. Then:*

$$\begin{array}{lll}
\mathbf{T} \models i \rightarrow \neg Fi & \textit{iff} & < \textit{ is irreflexive;} \\
\mathbf{T} \models i \rightarrow \neg FFi & \textit{iff} & < \textit{ is asymmetric;} \\
\mathbf{T} \models i \rightarrow G(Fi \rightarrow i) & \textit{iff} & < \textit{ is antisymmetric;} \\
\mathbf{T} \models Pi \vee i \vee Fi & \textit{iff} & < \textit{ is trichotomous;} \\
\mathbf{T} \models FPi & \textit{iff} & < \textit{ is right directed;} \\
\mathbf{T} \models i \rightarrow (F\top \rightarrow FH H\neg i) & \textit{iff} & < \textit{ is right discrete.}
\end{array}$$

Proof: Straightforward, and left to the reader. □

Indeed, with the aid of nominals we can define $\mathbf{Z}$, the integers in their usual order, up to isomorphism. We see this as follows. Let ϕ^{STO} be

$$(i \rightarrow \neg Fi) \wedge (Pi \vee i \vee Fi) \wedge (FFi \rightarrow Fi).$$

The first conjunct defines irreflexivity, the second trichotomy, and the third transitivity, thus ϕ^{STO} is valid on precisely the STOs. Now, as we have discussed, van Benthem has shown that the purely Priorean wff ϕ^{Z} defines $\mathbf{Z}$ on the class of STOs — hence $\phi^{STO} \wedge \phi^{Z}$ is valid on precisely those frames isomorphic to $\mathbf{Z}$. Summing up: nominals can see temporal structure that propositional variables are blind to. Sorting has *not* lead to syntactic sugar: nominal tense logic is a genuinely richer than Priorean tense logic.

Our first referential sort looks promising, so let's introduce another. Much work in both temporal knowledge representation (e.g. Allen [1]) and natural language semantics (e.g. Dowty [21]) insists on the importance of extended *periods of time*, or *intervals*. We will now introduce a third sort, the sort of *interval nominals*, and constrain their interpretation so that the set of points at which any interval nominal is true is an interval. Thus, even though we will continue to work with a point-based semantics, we will gain a partial handle on interval structure. For certain purposes, admittedly, this handle will turn out to be inadequate, and we'll eventually be forced to sort in the richer setting of interval-based semantics. Nonetheless, it's interesting to see how far we can go with simpler equipment.

So choose a third denumerably infinite set INOM, mutually disjoint from both NOM and VAR. We call the elements of INOM interval nominals, and typically represent them using e, d, c, and so on. Next we define ATOM to be VAR $\cup$ NOM $\cup$ INOM. Once again, this is the only syntactic change we make: wffs are made from this expanded set of atoms in the usual way.

What about the semantics? The following seems a plausible characterization of what we mean by an interval: *an interval is an unbroken stretch of time.* Let's make this precise. Let $\mathbf{T}$ $(= \langle T, < \rangle)$ be a frame. A *convex* subset of T is a subset S of T such that $(\forall s, s' \in S)(\forall t \in T)(s < t < s' \Rightarrow t \in S)$. That is, convex subsets are unbroken stretches of time. We shall insist that when interpreting our language on a frame $\mathbf{T} = \langle T, < \rangle$, for all interval nominals e, $V(e)$ must be a non-empty convex subset of T. As before, no constraints will be placed on the denotations of variables, and nominals must denote points. A function V : ATOM $\longrightarrow Pow(T)$ that respects these constraints is called a valuation, and a model is a pair $\langle \mathbf{T}, V \rangle$ where $\mathbf{T}$ is a frame and V a valuation on $\mathbf{T}$. Semantic concepts such as satisfaction and validity are defined as before. We call this three-sorted modal language *referential tense logic.*[11]

[11] Referential tense logic is a subsystem of the richer many-sorted languages defined in Blackburn [5, 8] that deal with clock-expressions, days, months, years, and temporal indexicals such as yesterday, today, tomorrow and now.

Now, in the following section we will use referential tense logic to model (parts of) Allen's system, and Allen views temporal precedence as linear. So what can we do with referential tense logic when working with STOs? The key observation is that over STOs the *universal modality* $\Box$ and its dual $\Diamond$ are definable, and both modalities interact elegantly with nominals and interval nominals. Now, $\Box\phi$ is true at a time t iff ϕ is true at *all* times. Over STOs this means we need can define $\Box\phi$ to be $H\phi \wedge \phi \wedge G\phi$. The dual operator $\Diamond$ is thus $\neg\Box\neg\phi$ (that is, $P\phi \vee \phi \vee F\phi$) and clearly $\Diamond\phi$ means that ϕ is true at *some* time. Now for the crucial point: these operators interact beautifully with our two referential sorts. For example $\Box(i \rightarrow \phi)$ (and indeed, $\Diamond(i \wedge \phi)$) can be regarded as a 'jump' instruction. Because nominals are true at exactly one point, these wffs are true iff the information ϕ holds at the point named by i. In effect, these wffs shift the point of evaluation to the point named by i and test for the truth of ϕ there. Matters are similar with interval nominals: $\Box(e \rightarrow \phi)$ is true iff the information ϕ holds at *all* points in the interval named by e. Such combinations are precisely what we shall use to model Allen's HOLDS predicate. Indeed if we wanted to work with nominals or interval nominals over *non*-linear time flows — on such flows $\Box$ cannot be defined in terms of the tense operators — it would be an excellent idea to introduce the universal modality as an additional primitive.[12]

So: we now have a simple looking bi-modal language in which reference to both points and intervals is possible. But has this increased expressivity come at a computational cost? Pleasantly enough, at least when working with linear time, the answer is *no*. I'm now going to show that the satisfiability problem for the referential tense logic of any class of STOed frames is no worse than the satisfiability problem for the Priorean tense logic of the same frame class. I'll do this by (polynomial time) reducing the former problem to the latter. At the heart of the following argument is the following observation: although the language of Priorean tense logic is not referential, it *is* possible (at least when working with linear time), to simulate temporal reference by making use of certain defined operators.

We've already met two of the defined operators we'll need, namely the universal modality $\Box$ and its dual $\Diamond$. In addition we'll use the *difference* operator

[12] In fact, virtually everyone who has worked with nominals has examined the effects of introducing the universal modality as a primitive, precisely because its combination with referential sorts is so effective. Indeed in Goranko and Passy [31], the standard reference on the universal modality, one of the main motivations given for being interested in the universal modality in the first place is because of its smooth interaction with nominals. A number of recent papers on hybrid languages (notably Areces, Blackburn and Marx [2], Blackburn [9], and Blackburn and Tzakova [16] have examined the consequences of introducing only the 'jump' operators noted above as primitive. That is, instead of adding a primitive universal modality, these papers add a primitive operator $@_i$ for each nominal i, where $@_i\phi$ means "jump to the point named by the nominal i and test for the truth of ϕ there". For some hybrid languages the use of primitive $@_i$ can be preferable to introducing a primitive universal modality (for example, on complexity theoretic grounds; see Areces, Blackburn and Marx [2] . Moreover, the $@_i$ operators support elegant proof theories (see Seligman [46], Blackburn [9] .

D. The difference operator is a modality that explores the inequality relation on frames. That is, $D\phi$ is true at a point t in a model iff ϕ is true at a point t' such that $t' \neq t$. Because we are working with STOed frames, $D\phi$ is definable: $P\phi \vee F\phi$ suffices.[13] With the help of these operators we can define operators U and I that will enable us to simulate referential sorts. Define $U\phi$ to be $\Diamond(\phi \wedge \neg D\phi)$. That is, $U\phi$ is true if ϕ is true at a *unique* point. The U operator gives us a way of simulating nominals. Define $I\phi$, to be $\Diamond\phi \wedge \Box(P\phi \wedge F\phi \rightarrow \phi)$. That is, $I\phi$ is true in a model iff ϕ is true somewhere, and, in addition, the set of points at which ϕ is true is convex. Thus I gives us a way of simulating interval nominals.

We now define, for any wff ϕ of referential tense logic, a corresponding purely Priorean wff $\phi^\flat$ with two pleasant properties. First, the length of $\phi^\flat$ is linear in the length of ϕ. Second, ϕ is satisfiable iff $\phi^\flat$ is. As a first step, let β be any bijection from VAR $\cup$ NOM $\cup$ INOM to VAR. We use β as the base clause for a simple substitution translation σ: $\sigma(a) = \beta(a)$, for all atoms a; $\sigma(\neg\phi) = \neg\sigma(\phi)$; $\sigma(\phi B \psi) = \sigma(\phi) B \sigma(\psi)$ for all binary Boolean connectives B; $\sigma(F\phi) = F\sigma(\phi)$; and $\sigma(P\phi) = P\sigma(\phi)$. Thus, given a wff of referential tense logic ϕ, $\sigma(\phi)$ is a purely Priorean wff that has exactly the same form as ϕ: all we have done is substitute propositional variables for the atoms in ϕ.

We are now ready to define $\phi^\flat$. Let $i_1, \ldots, i_n$ and $e_1, \ldots, e_m$ be the nominals and interval nominals that occur in ϕ. Then we define $\phi^\flat$ to be:

$$U(\sigma(i_1)) \wedge \cdots \wedge U(\sigma(i_n)) \wedge I(\sigma(e_1)) \wedge \cdots \wedge I(\sigma(e_m)) \wedge \sigma(\phi).$$

If ϕ contains no nominals or interval nominals we simply omit the relevant conjuncts from $\phi^\flat$. Thus when ϕ is purely Priorean, $\phi^\flat$ is just $\sigma(\phi)$.

Lemma 2.1 *For any wff ϕ of referential tense logic, $|\phi^\flat|$ is linear in $|\phi|$.*

Proof: Left to the reader. $\Box$

Lemma 2.2 *Let ϕ be any wff of referential tense logic, and $\mathbf{T}$ be any* STO. *Then ϕ is satisfiable in a $\mathbf{T}$-based model iff $\phi^\flat$ is satisfiable in a $\mathbf{T}$-based model.*

Proof: Let ϕ be a wff of referential tense logic, let $\mathbf{T}$ be a STO, let $\mathbf{M} = \langle \mathbf{T}, V \rangle$, and suppose that $\mathbf{M}, t \models \phi$. Let $V^\flat$ be any valuation such that $V^\flat(p) = V(\beta^{-1}(p))$ for all propositional variables p, and let $\mathbf{M}^\flat$ be the model $\langle \mathbf{T}, V^\flat \rangle$. $V^\flat$ assigns singletons and convex subsets to all propositional variables which are β images of nominals and interval nominals respectively. Thus for all nominals i, $U(\sigma(i))$ is globally true in $\mathbf{M}^\flat$, and for all interval nominals e, $I(\sigma(e))$ is globally true in $\mathbf{M}^\flat$. Clearly $\mathbf{M}^\flat, t \models \sigma(\phi)$. Thus $\mathbf{M}^\flat, t \models \phi^\flat$.

[13] Like the universal modality, the difference operator is now a standard tool in modal logic; de Rijke [44] is the best source for further information. The difference operator also plays an important role in the proof-theoretic analyses of nominal tense logic in Demri [19] and Demri and Goré [20].

Conversely, suppose there is a model $\mathbf{M}^\flat = \langle \mathbf{T}, V^\flat \rangle$ and a point t in $\mathbf{T}$ such that $\mathbf{M}^\flat, t \models \phi^\flat$. Let V be any function from ATOM to $Pow(\mathbf{T})$ such that (a) $V(\beta^{-1}(p)) = V^\flat(p)$ for all propositional symbols occurring in $\phi^\flat$, (b) $V(i)$ is a singleton, for all nominals i such that $\sigma(i)$ does not occur in $\phi^\flat$, and (c) $V(e)$ is a convex subset for all interval nominals e such that $\sigma(e)$ does not occur in $\phi^\flat$. V is a valuation: in particular, the relevant constraints hold for all nominals and interval nominals whose σ images *do* occur in $\phi^\flat$ because all conjuncts of the form $U(\sigma(i))$ and $I(\sigma(e))$ are true in $\mathbf{M}^\flat$. Thus $\mathbf{M} = \langle \mathbf{T}, V \rangle$ is a model, and clearly $\mathbf{M}, t \models \phi$. $\Box$

Theorem 2.2 *Let $\mathcal{S}$ be a class of* STO*s. Determining whether an arbitrary wff of referential tense logic is satisfiable in a model based on a frame in $\mathcal{S}$, is linear time reducible to the satisfiability problem for Priorean tense logic on $\mathcal{S}$.*

Proof: To test whether a wff ϕ of referential tense logic is satisfiable in a model based on a STO in $\mathcal{S}$, we need merely form $\phi^\flat$ (which by Lemma 2.1 we can do in linear time) and attempt to satisfy $\phi^\flat$ on some such model (which by Lemma 2.2 is a necessary and sufficient condition for the satisfiability of ϕ in such a model). But $\phi^\flat$ is purely Priorean. $\Box$

As a simple application of this result, we have that the satisfiability problem for the referential tense logic of $\mathbf{Q}$ $(= \langle Q, < \rangle)$, the rational numbers in their usual order, is NP-complete. We argue as follows. Clearly this problem is NP hard, as it contains the satisfiability problem for propositional calculus as a special case. On the other hand, the previous theorem tells us that the problem of satisfying Priorean wffs in $\mathbf{Q}$-based models gives us an upper bound for the problem, and from the work of Ono and Nakamura [38] we know that the satisfiability problem for Priorean wffs on $\mathbf{Q}$-based models is solvable in NP-time. Thus the NP-completeness result for referential tense logic follows.[14]

To close this section, some historical remarks and pointers to the literature on hybrid languages. Arthur Prior [40] introduced nominals into tense logic (he called them *world propositions*). Arthur Prior wasn't primary interested in making temporal *reference* possible, rather he was interested in *quantifying*

[14]Note that the proof of Theorem 2.2 hinges on the definability of U and I, which in turn hinges on the fact that $\Box$ and D are definable when working with STOs. If we were working on a class of frames in which these operators were not definable, then adding them as primitives *can* have consequences for computational complexity. For example, in Blackburn and Spaan [13] it is shown that the satisfiability problem for multimodal languages with nominals over deterministic frames is NP-complete. The universal modality is not definable over this class of frames, and if it as added as a primitive, the satisfiability problem becomes EXPTIME-complete. Furthermore, recent work has shown that over some classes of frames, the satisfiability problem for nominal tense logic can be more complex than the satisfiability problem for Priorean tense logic. For example, the satisfiability problem for Priorean tense logic over the class of all frames is PSPACE-complete, but the satisfiability problem for nominal tense logic is EXPTIME-complete; see Areces, Blackburn and Marx [2].

over times in a modal framework.[15] Thus in Prior's hybrid languages it was possible to bind nominals using $\exists$ and $\forall$. Robert Bull was the first to investigate sorting technically; his paper on the subject (Bull [17]) is an overlooked classic of temporal logic. Among other things, Bull introduced a three-sorted language: propositional variables, nominals, and a *course-of-history* sort (that is, atomic formulas constrained to be true at the points on some *path* through the frames; Bull's pioneering work predates by more than a decade the use of path-based temporal logics in computer science).[16] Bull allowed both nominals and course-of-history symbols to be bound using $\exists$ and $\forall$, and contained a primitive universal modality, thus his system had massive expressive power.

Following Prior's death in 1969, work on hybrid languages seems to have lain dormant till sorting was reinvented by the Sofia school in the 1980s. Their work concentrated on the use of nominals in propositional dynamic logic (PDL); Passy and Tinchev [39] is an encyclopedic discussion of the subject, and Gargov and Passy [25] show that nominals permit an elegant treatment of looping and determinism in PDL.

Although the Sofia School did interesting work on systems in which nominals could be bound by $\forall$ and $\exists$, one of their most important legacies was the attention they devoted to *weaker*, systems, a topic which has dominated work on hybrid languages in the 1990s.[17] A key paper on nominals in the setting of uni-modal logic (sometimes with the addition of a primitive universal modality) is Gargov and Goranko [26].[18] Blackburn [5] considers the effects of enriching Priorean tense logic with nominals, interval nominals and various other sorts.[19] Shortly thereafter, papers began to appear which considered hybrid languages based on novel primitives, notably the jump operators $@_i$ mentioned in Footnote 12, and a binder $\downarrow$ which binds nominals to the point of evaluation; see Goranko [28, 27] and Blackburn and Seligman [11, 12]. The recent Areces, Blackburn and Marx isolates the crucial systems and proves a number of fundamental expressivity, interpolation and complexity result. Over the same period there was intensive work on both Hilbert-style (Blackburn and Tzakova [14, 16]) and a wide range of non-Hilbert style proof systems (Seligman [46], Konikowska [37], Blackburn [9], Demri [19], Demri and Gore [20], and Tzakova [52]). Hybrid lan-

[15] Prior does not seem to have fully appreciated the importance of temporal reference, though he did write a paper which uses sorting to handle the temporal indexical now (see Prior [41] , and he sometimes used sorting to handle clock-expressions.

[16] Goranko [29] investigates standard path-based temporal logics using hybrid languages.

[17] Many tasks, such as proving general completeness results, are relatively easy for hybrid language containing $\forall$ and $\exists$; isolating well-behaved weaker systems is more demanding.

[18] This paper contains a beautiful expressivity result: as far as frame definability is concerned, it is irrelevant whether we enrich a modal language with the difference operator D, or with both nominals and the universal modality. One direction of this equivalence is obvious: as we saw above, the D operator is strong enough to simulate nominals, and clearly $\phi \wedge \neg D \neg \phi$ defines the universal modality. But the reverse direction is neither obvious nor easy to establish.

[19] The key results for nominals can be found in Blackburn [6], while interval nominals and other natural language inspired sorts sorts are discussed in Blackburn [8].

guages — especially the weaker systems — are now far better understood than they were in 1990.

One of the strengths of work on hybrid languages is that it has usually been guided by questions arising in concrete applications: Bull's work on paths was motivated by the need to define a strong future tense operator for branching time, the work of the Sofia school was influenced by issues by theoretical computer science, and my own early work on the subject was influenced by natural language semantics. This interplay between theory and practice continues to the present day. For example, the AVM notation used in computational linguistics turns out to be syntactic sugar for multi-modal logic enriched with nominals; see Blackburn [7] and Blackburn and Spaan [13] and Reape [43]. More recently it has become clear that hybrid languages offer a natural framework for discussing description (or terminological) logics; a preliminary overview can be found in Blackburn and Tzakova [15], but there is more recent unpublished work on this topic. Finally, modal logics of relative similarity and rough sets have recently been enriched with nominals (see Konikowska [37]). Thus the discussion of temporal knowledge representation which follows is very much in the spirit of a longstanding tradition.

3 A first look at Allen's system

Let's make our first attempt to bring together ideas from philosophical logic and AI: we'll attempt to capture the leading ideas of Allen's [1] theory of time in referential tense logic.

Allen's system

Allen's system is a quasi first-order language in which various relationships between intervals can be expressed, properties can be asserted to hold over intervals, and events and processes (for which Allen's blanket term is occurrences) can be said to occur at intervals.

Allen adopts a linear view of time, and he makes use of all thirteen possible relations that can hold between two intervals in an STO. These thirteen mutually exclusive binary relations are the equality relation $=$, and in addition B(efore), M(eets), O(verlaps), S(tarts), D(uring), and F(inishes) and their inverses, $\overline{B}$, $\overline{M}$, $\overline{O}$, $\overline{S}$, $\overline{D}$ and $\overline{F}$. The relations are best introduced by example. Consider the frame $\mathbf{Q}$, the rational numbers in their usual ordering. Then:

$$
\begin{array}{ll}
B([0,1],[2,3]) & \overline{B}([2,3],[0,1]) \\
M([0,1],(1,2]) & \overline{M}([1,2],[0,1)) \\
O([0,2],[1,3]) & \overline{O}([1,3],[0,2]) \\
S([0,1],[0,2]) & \overline{S}([0,2],[0,1]) \\
D([1,2],[0,3]) & \overline{D}([0,3],[1,2]) \\
F([1,2],[0,2]) & \overline{F}([0,2],[1,2])
\end{array}
$$

Allen has a two place relation symbol for talking about each of these relations, thus his language contains such items as BEFORE, MEETS, FINISHES and so on. Axiomatic constraints on the thirteen relations and their interrelationships are stated using this vocabulary. Two typical axioms are:

$$\mathrm{MEETS}(t_1,t_2) \wedge \mathrm{MEETS}(t_2,t_3) \rightarrow \mathrm{BEFORE}(t_1,t_3),$$

and

$$\begin{aligned}&\mathrm{MEETS}(t_1,t_2) \wedge \mathrm{DURING}(t_2,t_3) \rightarrow \\ &\qquad \mathrm{OVERLAPS}(t_1,t_3) \vee \mathrm{DURING}(t_1,t_3) \vee \mathrm{STARTS}(t_1,t_3)\end{aligned}$$

This should give the flavor of Allen's notation for talking about intervals. As is clear from these examples, Allen makes use of the standard Boolean connectives and variables over intervals. Moreover, he uses the quantifiers $\forall$ and $\exists$. In short, this part of his system is an orthodox first-order theory of interval structure.

Allen's then introduces *properties* and *occurrences*. Intuitively, properties P are those states of affairs expressed by such English predicates as is tattooed, is a killer and is covered in bl e

To deal with properties and events Allen introduces two further binary predicates, HOLDS and OCCUR. HOLDS takes a property symbol P and an interval term T, and the resulting expression HOLDS(P, T) asserts that the property P holds over the interval denoted by T. Similarly, OCCUR takes an event symbol E and an interval T, and the resulting expression OCCUR(E, T) asserts that the event E happened over the interval T. These two predicates are governed by axioms which reflect the intuitions about properties and events just discussed. First, Allen demands that:

$$\mathrm{OCCUR}(E, T) \ \& \ \mathrm{IN}(t, T) \Rightarrow \neg\mathrm{OCCUR}(E, t).$$

An event cannot occur at two intervals, one of which is a subinterval of the other. Second, he demands that:

$$\mathrm{HOLDS}(P, T) \Leftrightarrow (\forall t)[\mathrm{IN}(t, T) \Rightarrow \mathrm{HOLDS}(P, t)]$$

where IN(t, T) is defined as DURING$(t, T) \vee$ STARTS(t, T) $\vee$ FINISHES(t, T). The truth of a property sentence at an interval trickles down to all proper subintervals.[21]

So far so good. However Allen then goes on to elaborate his account of properties, and here matters become rather murky. The elaboration concerns the property terms: Allen allows property symbols to be combined using certain functions named *and*, *or* and *not*, and from some of the axioms Allen introduces to govern these functions, for example

$$\mathrm{HOLDS}(\mathit{and}(P, Q), T) \Leftrightarrow \mathrm{HOLDS}(P, T) \ \& \ \mathrm{HOLDS}(P, Q),$$

it's clear that he wants an 'internal logic' of property terms that mirrors the 'external logic' of formulas. Indeed, Allen also introduces functions *exists* and *all* which can bind variables in property terms. The intention is clear: Allen wants to treat properties as term-like entities that behave in a formula-like manner (and in fact, Allen explicitly says he wants properties to name complex logical expressions). But some of his axioms seem rather unmotivated. Moreover, the structure of property terms is never fully specified, and it remains unclear exactly what can and cannot be done in this part of his system.[22]

Nonetheless, Allen's theory has been one of the most influential accounts in AI of temporal representation and reasoning, and certainly its leading intuitions are natural. Can they be captured in referential tense logic? As we shall see, we can successfully capture the ideas that underly Allen's treatment of properties. We'll do so by treating referential tense logic as akin to a low level programming

[21] This is a simplification. Allen also introduces an stronger axiom, HOLDS$(P, T \ \Leftrightarrow$ $(\forall t$ [IN$(t, T \ \Rightarrow (\exists s$ [IN$(s, t \ \&$ HOLDS$(P, s$]]. His reasons for doing so aren't relevant to our discussion, and, as Galton [24] shows, this stronger axiom leads to difficulties.

[22] Similar criticisms of this aspect of Allen's work have been made by Turner [51] and Shoham [47].

language and define a set of high level macros over it. First we'll define analogs of Allen's interval-relating predicates such as MEETS and STARTS. This is easy to do using the referential sorts, and the macros we define automatically inherit the correct logical behavior from the semantics of referential tense logic. We then define an analog of the HOLDS predicate. This is where our sorting strategy *really* pays off. After all, in referential tense logic all types of information are treated in syntactically uniform fashion. In particular, referential information is represented using nominals and interval nominals, and these are *formulas*. Thus all types of information can be freely combined using the logical symbols. We simply don't need to add a separate 'internal logic' — a genuine logic is already hard-wired into the semantics of referential tense logic.

Simulating Allen's system

Our first task is to show that we can simulate Allen's interval relation symbols. As referential tense logic uses interval nominals to pick out intervals, this means we must show that given two interval nominals, say e and d, we can assert that any of the thirteen relations discussed above holds between their denotations. This is easily done. For any interval nominals e and d we define:

$$
\begin{array}{lll}
\mathbf{Equals}(e,d) & =_{def} & \Box(e \leftrightarrow d) \\
\mathbf{Before}(e,d) & =_{def} & \Box(e \rightarrow \neg d \wedge Fd) \wedge \Diamond(\neg e \wedge \neg d \wedge Pe \wedge Fd) \\
\mathbf{After}(e,d) & =_{def} & \Box(e \rightarrow \neg d \wedge Pd) \wedge \Diamond(\neg e \wedge \neg d \wedge Fe \wedge Pd) \\
\mathbf{Meets}(e,d) & =_{def} & \Box(e \rightarrow \neg d \wedge Fd) \wedge \neg\Diamond(\neg e \wedge \neg d \wedge Pe \wedge Fd) \\
\overline{\mathbf{Meets}}(e,d) & =_{def} & \Box(e \rightarrow \neg d \wedge Pd) \wedge \neg\Diamond(\neg e \wedge \neg d \wedge Fe \wedge Pd) \\
\mathbf{Overlaps}(e,d) & =_{def} & \Diamond(e \wedge d) \wedge \Diamond(e \wedge \neg d \wedge Fd) \wedge \Diamond(d \wedge \neg e \wedge Pe) \\
\overline{\mathbf{Overlaps}}(e,d) & =_{def} & \Diamond(e \wedge d) \wedge \Diamond(e \wedge \neg d \wedge Pd) \wedge \Diamond(d \wedge \neg e \wedge Fe) \\
\mathbf{Starts}(e,d) & =_{def} & \Box(e \rightarrow d) \wedge \Diamond(d \wedge \neg e) \wedge \Box(e \wedge d \rightarrow H(d \rightarrow e)) \\
\mathbf{Co\text{-}starts}(e,d) & =_{def} & \Box(d \rightarrow e) \wedge \Diamond(e \wedge \neg d) \wedge \Box(d \wedge e \rightarrow H(e \rightarrow d)) \\
\mathbf{During}(e,d) & =_{def} & \Box(e \rightarrow d) \wedge \Diamond(d \wedge \neg e \wedge Pe) \wedge \Diamond(d \wedge \neg e \wedge Fe) \\
\mathbf{Contains}(d,e) & =_{def} & \Box(d \rightarrow e) \wedge \Diamond(e \wedge \neg d \wedge Pd) \wedge \Diamond(d \wedge \neg e \wedge Fd) \\
\mathbf{Finishes}(e,d) & =_{def} & \Box(e \rightarrow d) \wedge \Diamond(d \wedge \neg e) \wedge \Box(e \wedge d \rightarrow G(d \rightarrow e)) \\
\mathbf{Co\text{-}finishes}(e,d) & =_{def} & \Box(d \rightarrow e) \wedge \Diamond(e \wedge \neg d) \wedge \Box(d \wedge e \rightarrow G(e \rightarrow d))
\end{array}
$$

Though somewhat ungainly, these definitions have the required effect. For example, consider the definition of $\mathbf{Starts}(e,d)$. The first conjunct asserts that the interval named by e is a subinterval of the interval named by d. The second conjunct guarantees the existence of a time where d is true and e is not, thus the force of the first two conjuncts is that the interval named by e is a proper subinterval of that named by d. Now consider the third conjunct. This says that at any time where both e and d are true, then at all previous times, if d is true, e is true also. To put it another way, we can't find any time t where both e and d are true and such that at some earlier time t' we have that d holds and

e doesn't. That is, both intervals must start at the same time, and thus the net affect of the three conjuncts captures what is required.

So we can define expressions that correspond to Allen's predicates. Moreover, the logic of these expressions is the logic desired by Allen. For example, consider Allen's axiom

$$\text{MEETS}(t_1, t_2) \wedge \text{MEETS}(t_2, t_3) \rightarrow \text{BEFORE}(t_1, t_3).$$

The macro

$$\mathbf{Meets}(e, d) \wedge \mathbf{Meets}(d, c) \rightarrow \mathbf{Before}(e, c),$$

is a formal analog of this. And this macro is valid: expanding it into referential tense logic yields:

$$\begin{aligned}&\Box(e \rightarrow \neg d \wedge Fd) \wedge \neg\Diamond(\neg e \wedge \neg d \wedge Pe \wedge Fd) \wedge \\ &\Box(d \rightarrow \neg c \wedge Fc) \wedge \neg\Diamond(\neg d \wedge \neg c \wedge Pd \wedge Fc) \\ &\qquad \rightarrow \Box(e \rightarrow \neg c \wedge Fc) \wedge \Diamond(\neg e \wedge \neg c \wedge Pe \wedge Fc),\end{aligned}$$

which cannot be falsified.

If we want, we can extend these macros to take nominals as well as interval nominals. That is, we could allow ourselves to write down such expressions such as $\mathbf{Before}(i, j)$, and $\mathbf{Starts}(i, e)$, where i and j are nominals. $\mathbf{Before}(i, j)$, which is just

$$\Box(i \rightarrow \neg j \wedge Fj) \wedge \Diamond(\neg j \wedge \neg i \wedge Pi \wedge Fj)$$

says that that point named by i is properly before that named by j; while $\mathbf{Starts}(i, e)$, which expands into

$$\Box(i \rightarrow e) \wedge \Diamond(e \wedge \neg i) \wedge \Box(i \wedge e \rightarrow H(e \rightarrow i)),$$

says that that point named by i is the earliest point in the interval named by j.

With the first task accomplished, let's turn to the second: mimicking the way Allen's system deals with properties. Can we define a macro **Holds** that has the properties Allen desires of HOLDS? Such a macro must mirror the downwards persistence property:

$$\text{HOLDS}(P, T) \Leftrightarrow (\forall t)[IN(t, T) \Rightarrow \text{HOLDS}(P, t)],$$

and furthermore must possess a suitable 'internal logic'.

The required macro could hardly be simpler. For any wff ϕ and any interval nominal e we define:

$$\mathbf{Holds}(\phi, e) \;\; =_{def} \;\; \Box(e \rightarrow \phi).$$

In short, **Holds** is just the canonical combination of universal modality and referential sort noted in the previous section. It's clear that it behaves appropriately. First, it satisfies downwards persistence. To see this, let $\mathbf{In}(e, d)$ be defined to hold iff $\mathbf{During}(e,d) \vee \mathbf{Starts}(e,d) \vee \mathbf{Finishes}(e,d)$. A little thought reveals that $\mathbf{In}(e,d)$ holds iff $\Box(e \rightarrow d) \wedge \neg\mathbf{Equals}(d,e)$, and thus (as we would hope) the **In** macro expresses that the interval named by e is a proper subinterval of that named by d. Then for any any interval nominals e and d and any wff ϕ we have that the following expression is valid:

$$\mathbf{Holds}(\phi, e) \wedge \mathbf{In}(d, e) \rightarrow \mathbf{Holds}(\phi, d).$$

We also inherit from the semantics of referential tense languages an internal logic of **Holds** analogous to the internal logic of HOLDS that Allen seems to want. For example we have that all instances of

$$\mathbf{Holds}(\neg\phi, e) \rightarrow \neg\mathbf{Holds}(\phi, e)$$

and

$$\mathbf{Holds}(\phi \wedge \psi, e) \leftrightarrow \mathbf{Holds}(\phi, e) \wedge \mathbf{Holds}(\psi, e)$$

are valid. In passing, note that we can extend the use of **Holds** to nominals: $\mathbf{Holds}(\phi, i)$, which is just $\Box(i \rightarrow \phi)$ states that the the property ϕ holds at the point picked out by i. But the main point is this: we *don't* need to try to "construct" a logic of property terms by introducing new logical functions and trying to constrain them with suitable axioms. Everything we need is already in place.

To sum up, referential tense logic captures Allen's calculus of properties, and does so fairly cleanly. Firstly there's the (fairly obvious) point that because we can refer to intervals in this language, we can simulate Allen's predicates. Secondly there's the rather more subtle (and far more important) point concerning syntactical uniformity: because *all* types of information, including referential information, is handled using formulas, we didn't have to introduce 'logical functions'; the ordinary connectives sufficed.

But now that we've seen what can be simulated, let's turn to what *can't*, namely Allen's treatment of events. At first sight there seems to be an appealing way to to model events in referential tense logic. Until now we've used sorting as a tool for achieving temporal reference, but surely there are more obvious ways of using it? In particular, up till now we've treated ordinary propositional variables as placeholders for arbitrary information: properties and events are implicitly lumped together as things to be represented using propositional variables. But why not enforce sortal constraints here as well? For example, we could subdivide the variables VAR into two syntactically distinct sorts PVAR and EVAR. The elements of the former sort would be taken as ranging over primitive property sentences, and the latter over primitive event sentences. By imposing appropriate constraints on valuations we could try to capture the distinction and thus model the full scope of Allen's work in a four-sorted version of Priorean tense logic.

Nice idea, but it won't work. It's impossible to state constraints on tense logical valuations that capture Allen's notion of event: valuations in a point-based setting automatically enforce property-like behavior. Suppose q is an element of EVAR, that is to say, a variable earmarked to represent events. We want to say that q can never be true at two intervals t and t' such that one is a proper subinterval of the other. However in Priorean tense logic, saying that q is true over an interval t amounts to saying that q is true over all points in that interval. This means that if t is not a singleton interval, then q is true over *all* proper subintervals t' of t. Unless we restrict ourselves to the special case of point-events, no constraint on valuations is going to give us what we want: q cannot help but be some sort of property. The point-based semantics underlying referential tense logic tense logic is perfect for properties, and this very perfection renders it disastrous for events.

But the statement of the problem makes the remedy clear. The difficulty has nothing to do with modal languages, nor with the idea of sorting: it's simply that a point-based temporal semantics is too impoverished to support distinctions we wish to draw. So let's use a more appropriate semantic setting. In the following section we'll take as our base a certain interval-based language. Sorting this will give us what we want.

4 A sorted interval-based language

Interval-based languages, like languages of Priorean tense logic, are modal languages: they contain sentence operators reminiscent of those in Priorean logic, and adopt the internal view of time inherent in Kripke semantics. The crucial difference is that *wffs are evaluated at intervals.* This enables more sophisticated temporal distinctions to be drawn, and in particular, it's going to allow us to model occurrences. Actually, it will improve matters in other ways as well. The language we'll sort is that of Halpern and Shoham[47]. This is the interval-based language that naturally arises by abstracting from Allen's formalism, so our sortal analysis of Allen's work will be far neater than in referential tense logic.

The language of Halpern and Shoham contains: a denumerably infinite collection of propositional variables VAR, whose elements we write as p, q, r and so on, Boolean connectives, the punctuation symbols, and the one place sentence operators $\langle A\rangle$, $\langle\overline{A}\rangle$, $\langle B\rangle$, $\langle\overline{B}\rangle$, $\langle E\rangle$ and $\langle\overline{E}\rangle$. These deal with six of the fundamental relations between intervals in linear time. $\langle A\rangle$ picks out an interval that begins immediately after the current one, while $\langle\overline{A}\rangle$ picks out an interval that finishes immediately beforehand. $\langle B\rangle$ picks out an interval contained in the current one that begins when the current interval begins, while $\langle\overline{B}\rangle$ picks out at interval of which the current one is a beginning. $\langle E\rangle$ and $\langle\overline{E}\rangle$ are analogous to $\langle B\rangle$ and $\langle\overline{B}\rangle$ respectively, but pick out endings instead of beginnings. As we shall see, operators dealing with Allen's other relations are definable using these primitives.

Let's sort this language to make it referential. We'll add two referential sorts: SNOM, the *stretched interval nominals*, and PNOM, the *point-interval nominals*. (To simplify the terminology a bit, we'll usually refer to stretched interval nominals simply as *interval nominals*, and call point-interval nominals *point nominals*.) Stretched interval nominals will be used to name genuinely extended time periods — 'stretched out' periods of time — while point interval nominals will denote certain special point-like intervals. We assume that SNOM and PNOM are denumerably infinite sets, that VAR, SNOM and PNOM are mutually disjoint, represent the elements of SNOM as e, d, c, and those of PNOM by i, j, k. We'll call this language *referential interval logic*.[23]

The ingredients are in the bowl: now to mix them. Define ATOM to be VAR $\cup$ SNOM $\cup$ PNOM, and let WFF, the set of well formed formulas (or sentences) of our language be the smallest set containing ATOM that is closed under the application of the Boolean and interval operators. We introduce the following defined operators: $\langle D\rangle\phi =_{def} \langle B\rangle\langle E\rangle\phi$, $\langle L\rangle\phi =_{def} \langle A\rangle\langle A\rangle\phi$, $\langle O\rangle\phi =_{def} \langle E\rangle\langle\overline{B}\rangle\phi$, $\langle\overline{D}\rangle\phi =_{def} \langle\overline{B}\rangle\langle\overline{E}\rangle\phi$, $\langle\overline{L}\rangle\phi =_{def} \langle\overline{A}\rangle\langle\overline{A}\rangle\phi$ and $\langle\overline{O}\rangle\phi =_{def} \langle B\rangle\langle\overline{E}\rangle\phi$. As should be clear from the informal readings of the primitive operators, these defined operators deal with Allen's other relations. For example, $\langle D\rangle$ picks out an interval during the current one, while $\langle O\rangle$ picks out a 'future overlapping' interval. We define the duals $[A]$, $[\overline{A}]$, $[B]$, $[\overline{B}]$, $[E]$, $[\overline{E}]$, $[D]$, $[\overline{D}]$, $[L]$, $[\overline{L}]$, $[O]$ and $[\overline{O}]$, in the usual way. For example, $[\overline{E}]\phi =_{def} \neg\langle\overline{E}\rangle\neg\phi$.

Now for the semantics. Let $\mathbf{T}$ $(= \langle T, <\rangle)$ be an STOed frame (in the usual sense of tense logic) and let $I(\mathbf{T})$ be the set of all closed intervals $[t, t']$ $(= \{s \in T : t \leq s \leq t'\})$ on $\mathbf{T}$. The intervals in $I(\mathbf{T})$ fall naturally into two disjoint subclasses: $S(\mathbf{T})$ and $P(\mathbf{T})$. $S(\mathbf{T})$ is the set of all those intervals $[t, t']$ such that $t \neq t'$. We call $S(\mathbf{T})$ the *stretched intervals* on $\mathbf{T}$; they model extended and connected stretches of time, or proper intervals. $P(\mathbf{T})$ is the set of intervals $[t, t']$ such that $t = t'$; we call $P(\mathbf{T})$ the point-intervals on $\mathbf{T}$. Clearly such intervals are point-like.[24]

Given an STO $\mathbf{T}$, a valuation on $\mathbf{T}$ is a function V with domain ATOM and range $Pow(I(\mathbf{T}))$ that satisfies two constraints. First, for all stretched interval nominals e we insist that $V(e)$ is a singleton, and that if $[t, t']$ is the unique element in $V(e)$, then $[t, t'] \in S(\mathbf{T})$. That is, each interval nominal must denote a unique stretched interval. Second, for all point nominals i we insist that $V(i)$ is a singleton, and that if $[t, t']$ is the unique element in $V(i)$, then $[t, t'] \in P(\mathbf{T})$. That is, each point nominal must denote a unique point interval.[25]

[23] An aside. Referential interval logic is technically well behaved. In particular, the referential sorts we have introduced blend naturally with the analysis of Halpern and Shoham's system given in Venema [54], for nominals provide a natural tool for exploiting "rules for the undefinable" similar to those used by Venema.

[24] They're in 1-1 correspondence with the points of the underlying frame, and when $P(\mathbf{T}$ is equipped with the natural ordering $\prec$ defined by $[t, t] \prec [t', t']$ iff $t < t'$, the resulting frame is isomorphic to the original.

[25] Note that we're handling the distinction between interval nominals and point nominals

A model $\mathbf{M}$ is a pair $\langle \mathbf{T}, V\rangle$ where $\mathbf{T}$ is an STOed frame and V a valuation on $\mathbf{T}$ of the kind just described. We now define what it is for a model $\mathbf{M}$ $(= \langle T, <, V\rangle)$ to satisfy a wff ϕ at an interval $[t_1, t_2] \in I(\mathbf{T})$:

$$\begin{array}{lll}
\mathbf{M}, [t_1, t_2] \models a & \text{iff} & [t_1, t_2] \in V(a), \text{ for all atoms } a \\
\mathbf{M}, [t_1, t_2] \models \neg\phi & \text{iff} & \mathbf{M}, [t_1, t_2] \not\models \phi \\
\mathbf{M}, [t_1, t_2] \models \phi \wedge \psi & \text{iff} & \mathbf{M}, [t_1, t_2] \models \phi \text{ and } \mathbf{M}, [t_1, t_2] \models \psi \\
\mathbf{M}, [t_1, t_2] \models \langle A\rangle\phi & \text{iff} & \exists t_3 : t_2 < t_3 \text{ and } \mathbf{M}, [t_2, t_3] \models \phi \\
\mathbf{M}, [t_1, t_2] \models \langle \overline{A}\rangle\phi & \text{iff} & \exists t_3 : t_3 < t_1 \text{ and } \mathbf{M}, [t_3, t_1] \models \phi \\
\mathbf{M}, [t_1, t_2] \models \langle B\rangle\phi & \text{iff} & \exists t_3 : t_3 < t_2 \text{ and } t_1 \leq t_3 \text{ and } \mathbf{M}, [t_1, t_3] \models \phi \\
\mathbf{M}, [t_1, t_2] \models \langle \overline{B}\rangle\phi & \text{iff} & \exists t_3 : t_2 < t_3 \text{ and } \mathbf{M}, [t_1, t_3] \models \phi \\
\mathbf{M}, [t_1, t_2] \models \langle E\rangle\phi & \text{iff} & \exists t_3 : t_1 < t_3 \text{ and } t_3 \leq t_2 \text{ and } \mathbf{M}, [t_3, t_2] \models \phi \\
\mathbf{M}, [t_1, t_2] \models \langle \overline{E}\rangle\phi & \text{iff} & \exists t_3 : t_3 < t_1 \text{ and } \mathbf{M}, [t_3, t_2] \models \phi
\end{array}$$

A wff ϕ is *globally true* in $\mathbf{M}$ iff $\mathbf{M}, [t_1, t_2] \models \phi$ for all $[t_1, t_2] \in I(\mathbf{T})$, and in such a case we write $\mathbf{M} \models \phi$. A wff ϕ is *valid on a frame* $\mathbf{T}$ iff for all valuations V on $\mathbf{T}$, $\langle \mathbf{T}, V\rangle \models \phi$, and in such a case write $\mathbf{T} \models \phi$. If a wff ϕ is valid on all (STOed) frames we say it's *valid* and write $\models \phi$. If ϕ and ψ are wffs such that $\models \phi \leftrightarrow \psi$ then we say that ϕ and ψ are *logically equivalent*.

Let's consider the language more closely. First note that we can define interval analogs of the Priorean operators: for example, stipulating that $F\phi =_{def} \langle A\rangle\phi \vee \langle A\rangle\langle A\rangle\phi$, yields a future tense operator. Next, note that because we're working with a linear conception of time the language is strong enough to define the universal modality. We first define its dual:

$$\begin{array}{lll}
\Diamond\phi & =_{def} & \langle A\rangle\phi \vee \langle \overline{A}\rangle\phi \vee \langle B\rangle\phi \vee \langle \overline{B}\rangle\phi \vee \langle E\rangle\phi \vee \langle \overline{E}\rangle\phi \vee \\
& & \langle L\rangle\phi \vee \langle \overline{L}\rangle\phi \vee \langle D\rangle\phi \vee \langle \overline{D}\rangle\phi \vee \langle O\rangle\phi \vee \langle \overline{O}\rangle\phi \vee \phi.
\end{array}$$

That is, a wff $\Diamond\phi$ is true iff it is true at an interval related in one of the thirteen fundamental ways to the interval of evaluation. As any two intervals on an STOed frame must be related in one of these ways, we are entitled to read $\Diamond$ as 'at some interval'. We then define $\Box\phi$ to be $\neg\Diamond\neg\phi$, and read $\Box$ as 'at all intervals'. $\Box$ and $\Diamond$ co-operate with interval nominals and point nominals in the interval-based semantics just as well as they did in the point-based setting, and if we were working with intervals over non-linear time, or with a weaker choice of interval operators in which $\Box$ and $\Diamond$ were not definable, it would be a excellent idea to introduce them as primitives.

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slightly differently from the way did in referential tense logic. In referential tense logic interval nominals could name points, for singleton subsets of STOs are convex, and hence, under our previous definition, are intervals. However, in referential *interval* logic, interval nominals cannot name points, for $S(\mathbf{T}$ and $P(\mathbf{T}$ are disjoint. If desired, we could add a third referential sort to the language — call it INOM — whose members could refer to any interval whatsoever in $I(\mathbf{T}$. Such a sort would be analogous to the interval nominals of referential tense logic. However we'll stick with SNOM and PNOM in this paper.

We can also define the D operator: we only need to drop that last disjunct from the definition of $\Diamond$:

$$\begin{aligned} D\phi \quad =_{def} \quad & \langle A\rangle\phi \vee \langle \overline{A}\rangle\phi \vee \langle B\rangle\phi \vee \langle \overline{B}\rangle\phi \vee \langle E\rangle\phi \vee \langle \overline{E}\rangle\phi \vee \\ & \langle L\rangle\phi \vee \langle \overline{L}\rangle\phi \vee \langle D\rangle\phi \vee \langle \overline{D}\rangle\phi \vee \langle O\rangle\phi \vee \langle \overline{O}\rangle\phi. \end{aligned}$$

Clearly $D\phi$ holds at an interval iff ϕ holds at some other interval.

As with referential tense logic, the definability of D gives us some information about the complexity induced by our sorting. As before, we first use $\Diamond$ and D to define an operator U that insists that ϕ holds at a unique interval: $\Diamond(\phi \wedge \neg D\phi)$ suffices. Next we must simulate the point and the interval nominals. Observe that for any frame $\mathbf{T}$, $[B]\bot$ is true on precisely the elements of $P(\mathbf{T})$, while $\langle B\rangle\top$ is true on precisely the elements of $S\mathbf{T}$. So, define $U^p\phi$ to be of $U(\phi \wedge [B]\bot)$. $U^p\phi$ insists that ϕ is true at a unique *point* interval, and it enables us to simulate the point nominals. On the other hand, if we define $U^s\phi$ to be $U(\phi \wedge \langle B\rangle\top)$, we have a way of insisting that ϕ is true at a unique stretched interval, and this enables us to simulate interval nominals. So, by making use of U^p and U^s we can prove an interval-based analog of Theorem 2.2. We proceed pretty much as in our discussion of the complexity of referential tense logic. Define β to be a bijection from VAR $\cup$ SNOM $\cup$ INOM to VAR, and extend this to a substitution translation σ which commutes with all the operators. Then, for any wff ϕ, define $\phi^\flat$ to be:

$$U^p(\sigma(i_1)) \wedge \cdots \wedge U^p(\sigma(i_n)) \wedge U^s(\sigma(e_1)) \wedge \cdots \wedge U^s(\sigma(e_m)) \wedge \sigma(\phi),$$

where $i_1, \ldots, i_n$ and $e_1, \ldots, e_m$ are all the point nominals and interval nominals that occur in ϕ. Analogs of Lemma 2.1 and Lemma 2.2 are easily proved, and this leads to:

Theorem 4.1 *Let $\mathcal{S}$ be a class of* STO*s. Determining whether an arbitrary wff of referential interval logic is satisfiable in a model based on a frame in $\mathcal{S}$, is linear time reducible to the satisfiability problem for the system bei*[illegible]

$$\begin{array}{lll}
\mathbf{Equals}(r,r') & =_{def} & \Box(r \leftrightarrow r') \\
\mathbf{Before}(r,r') & =_{def} & \Box(r \rightarrow \langle L\rangle r') \\
\mathbf{After}(r,r') & =_{def} & \Box(r \rightarrow \langle \overline{L}\rangle r') \\
\mathbf{Meets}(r,r') & =_{def} & \Box(r \rightarrow \langle A\rangle r') \\
\overline{\mathbf{Meets}}(r,r') & =_{def} & \Box(r \rightarrow \langle \overline{A}\rangle r') \\
\mathbf{Overlaps}(r,r') & =_{def} & \Box(r \rightarrow \langle O\rangle r') \\
\overline{\mathbf{Overlaps}}(r,r') & =_{def} & \Box(r \rightarrow \langle \overline{O}\rangle r') \\
\mathbf{Starts}(r,r') & =_{def} & \Box(r \rightarrow \langle \overline{B}\rangle r') \\
\mathbf{Co\text{-}starts}(r,r') & =_{def} & \Box(r \rightarrow \langle B\rangle r') \\
\mathbf{During}(r,r') & =_{def} & \Box(r \rightarrow \langle \overline{D}\rangle r') \\
\mathbf{Contains}(r,r') & =_{def} & \Box(r \rightarrow \langle D\rangle r') \\
\mathbf{Finishes}(r,r') & =_{def} & \Box(r \rightarrow \langle \overline{E}\rangle r') \\
\mathbf{Co\text{-}finishes}(r,r') & =_{def} & \Box(r \rightarrow \langle E\rangle r')
\end{array}$$

The effect of these definitions should be clear: if both r and r' are instantiated in interval nominals, the resulting expression makes the same assertion about the two named intervals as does the use of Allen's corresponding predicate. For example, $\mathbf{Before}(e,d)$ is true in any model $\mathbf{M}$ iff the unique stretched interval denoted by e properly precedes that denoted by d. Note that if we instantiate at least one of r and r' in point nominals, the macros extend Allen's terminology to express relationships between a point and an interval, or between two points. Of course, not all of the expressions are meaningful in such cases, but all the incorrect uses are weeded out by the semantics. For example, $\mathbf{Overlaps}(i,j)$ is logically equivalent to $\perp$; our semantics (quite correctly) insists that we are talking nonsense.

Just as with referential tense logic, the force of Allen's axioms is captured by the semantics of our language. For example, corresponding to Allen's axiom:

$$\begin{array}{l}
\mathrm{MEETS}(t_1,t_2) \wedge \mathrm{DURING}(t_2,t_3) \rightarrow \\
\qquad \mathrm{OVERLAPS}(t_1,t_3) \vee \mathrm{DURING}(t_1,t_3) \vee \mathrm{STARTS}(t_1,t_3),
\end{array}$$

we can write down macros of the form

$$\mathbf{Meets}(e,d) \wedge \mathbf{During}(d,c) \rightarrow \mathbf{Overlaps}(e,c) \vee \mathbf{During}(e,c) \vee \mathbf{Starts}(e,c).$$

Expanding such a macro yields:

$$\Box(e \rightarrow \langle A\rangle d) \wedge \Box(d \rightarrow \langle \overline{D}\rangle c) \rightarrow \Box(e \rightarrow \langle O\rangle c) \vee \Box(e \rightarrow \langle \overline{D}\rangle c) \vee \Box(e \rightarrow \langle \overline{B}\rangle c),$$

which is easily seen to be valid.

Let's consider the HOLDS predicate. This can be mimicked as follows:

$$\mathbf{Holds}(\phi,e) =_{def} \Box(e \rightarrow \phi \wedge [D]\phi \wedge [B]\phi \wedge [E]\phi).$$

That is, just an in referential tense logic the definition of **Holds** hinges on the universal modality cooperating with the referential sorts. But note the crucial difference. In the point-based setting, information that holds over an interval *automatically* trickles down to sub-intervals (this is what blocked a treatment of occurrences). In the interval-based setting this doesn't happen, and if we want **Holds** to enforce information percolation down to sub-intervals, we have to spell out this requirement explicitly — hence the last three conjuncts in the definition's consequent.

Actually, we can do better than this. Galton [24] argues that Allen really requires not one but *three* variants of the HOLDS predicate: HOLDS-AT(P, I), which asserts that the property P holds at the instant I, HOLDS-IN(P, T), which asserts that the property P holds at some instant during the interval T, and HOLDS-ON(P, T), which asserts that the property P holds at every instant during the interval T. Discussing Galton's reasons for this claim would take us too far afield, but it's worth pointing out that referential interval logic can cope with Galton's extensions. For a start, it's clear how to simulate the HOLDS-AT operator: $\textbf{Holds-at}(\phi, i) =_{def} \Box(i \rightarrow \phi)$. What about HOLDS-IN and HOLDS-ON? Suppose we had the following 'point finding' operator $\langle\dagger\rangle$ at our disposal:

$$\langle \mathbf{T}, V\rangle, [t', t''] \models \langle\dagger\rangle\phi \text{ iff } (\exists [t,t] \in P(\mathbf{T}))([t,t] \subseteq [t', t''] \;\&\; \langle \mathbf{T}, V\rangle, [t,t] \models \phi).$$

We could then simulate the HOLDS-IN predicate as follows:

$$\textbf{Holds-in}(\phi, e) =_{def} \Box(e \rightarrow \langle\dagger\rangle\phi),$$

and, using the dual operator $[\dagger]$ of $\langle\dagger\rangle$, we could mimic HOLDS-ON:

$$\textbf{Holds-on}(\phi, e) =_{def} \Box(e \rightarrow [\dagger]\phi).$$

And in fact, $\langle\dagger\rangle$ *is* available as a defined operator. First (as Halpern and Shoham observe), we can define a 'beginning point' operator $[\![BP]\!]$ as follows:

$$[\![BP]\!]\phi =_{def} ([B]\bot \wedge \phi) \vee \langle B\rangle([B]\bot \wedge \phi).$$

That is, $[\![BP]\!]\phi$ is true at an interval t iff ϕ is true at the point that begins t. In similar fashion, we can define an 'end point' operator:

$$[\![EP]\!]\phi =_{def} ([E]\bot \wedge \phi) \vee \langle E\rangle([E]\bot \wedge \phi).$$

Using these we can define $\langle\dagger\rangle\phi$ as follows:

$$\langle\dagger\rangle\phi =_{def} [\![BP]\!]\phi \vee [\![EP]\!]\phi \vee \langle D\rangle[\![BP]\!]\phi.$$

Let's turn to what defeated us in the point-based setting: occurrences, and in particular, events. As a first step, let's define an analog of Allen's OCCUR. This is straightforward:

$$\textbf{Occur}(\phi, e) =_{def} \Box(e \rightarrow \phi \wedge [D]\neg\phi \wedge [B]\neg\phi \wedge [E]\neg\phi).$$

This is simply a variant of HOLDS in which percolation of information down to proper sub-intervals is explicitly *forbidden.*

There is one other thing that really should be done — and this concerns not merely occurrences, but properties as well. In the previous section we suggested imposing sortal constraints on the propositional variables to mirror the difference between properties and events. We couldn't do this, however, because the point-based semantics automatically enforced property-like behavior. Not only *can* we now do this, it is practically forced on us: without some constraints on valuations we have no guarantee that *any* of our propositional variables behave like properties or events! The information distributions possible in interval-based semantics are truly wild, and if we don't want the blooming buzzing confusion of a Humean universe we must impose order ourselves.

So let's do this. Subdivide VAR into two mutually disjoint and denumerably infinite sets, PVAR and EVAR. We call the elements of PVAR *property variables* and we'll represent its elements by p, p_1, p_2, p_3 and so on, and we'll call the elements of EVAR *event variables* and represent them by q, q_1, q_2, q_3 and so on. The definition of our language is unchanged: we have precisely the same atoms as before — VAR $\cup$ INOM $\cup$ SNOM — and we build the wffs out of them just as we used to.

Now for the semantics. Given an STOed frame $\mathbf{T}$, by a *PE-sorted valuation* V on $\mathbf{T}$ is meant a valuation that satisfies two further constraints.

$$(\forall p \in \mathrm{PVAR})(\forall \mathbf{i} \in I(\mathbf{T}))(\mathbf{i} \in V(p) \Leftrightarrow (\forall \mathbf{i}' \in I(\mathbf{T})(\mathbf{i}' \subset \mathbf{i} \Rightarrow \mathbf{i}' \in V(p))))$$

$$(\forall q \in \mathrm{EVAR})(\forall \mathbf{i}, \mathbf{i}' \in I(\mathbf{T}))(\mathbf{i} \in V(q) \ \& \ \mathbf{i}' \in V(q) \ \& \ \mathbf{i} \neq \mathbf{i}' \Rightarrow \mathbf{i} \not\subset \mathbf{i}'))$$

We then define *PE-sorted models* to be models whose valuations are PE-sorted. The satisfaction definition, and all the usual semantic definitions are exactly as before. This sorted version of Halpern and Shoham's logic captures the major ideas underlying Allen's framework: it can refer to times, reflects Allen's major ontological division (between properties and events), and is strong enough to deal with the refinements of HOLDS suggested by Galton.

5 Concluding remarks

In this paper I have argued that the gap between temporal logic in AI and philosophical logic is not as wide as at first appears, and that natural language provides an interesting bridge between the two traditions. Underpinning our discussion was the idea of sorted modal languages, or hybrid languages, as they are usually called.

Hybrid languages are based on a deceptively simple insight due to Arthur Prior: *many types of information can be viewed as propositions and manipulated using Boolean operators and modalities.* Not only does this enable us to draw some useful sortal distinctions (such as the distinction between properties and events) it also enables us to handle certain intrinsically *logical* ideas, notably those concerning temporal reference and the identity of times, which at first seem to lie beyond the reach of propositional formalisms. Moreover, the "democratic" treatment of information underlying Prior's approach enables us to treat temporal representation in a very straightforward way. When we discussed Allen's account of properties, it was clear that Allen wanted to treat properties as term-like entities that behaved in a formula-like manner. This is a powerful intuition — after all, it's all just information, and it seems we *should* be able to use it as we see fit. But Allen's first-order syntax with its formula/term distinction is a straitjacket that blocks the natural expression of this idea, and he was forced to introduce 'logical functions' to try and capture it. Prior's approach zeroes in on the crucial point: all there is is different sorts of temporal information, just waiting to be combined, and information can be viewed propositionally. There's simply no need to construct a second level of logic: right from the start, all types of information, referential and otherwise, play a first-class role in the logical economy.

Prior's idea meshes neatly with contemporary perspectives on modal logic. If you ask a modal logician why modal logic is interesting, it's unlikely that he or she will appeal to a particular application (such as reasoning about knowledge) to justify it. Rather, much interest in modal logic now centers on the fact that (propositional) modal languages are simple systems which correspond to interesting fragments of classical logic: first-order logic when talking about models, and second-order logic when talking about frames.[27] Effort is devoted to developing well behaved formalisms that offer interesting expressive power; work on the difference operator, for example, falls into this tradition. Hybrid languages are part of this general research program — but instead of focusing on new operators (such as D) the idea is to investigate what happens when the atomic propositions encode semantic distinctions. As we have seen, hybridization can lead to new expressive possibilities, and enables us to do interesting temporal modeling in a propositional language.

Sometimes, of course, we want something more powerful: in particular, we want to *quantify* across temporal entities. Certainly Allen wanted to be able to this; indeed (as we mentioned earlier) he even introduced a 'logical function' *exists* so that his internal logic could do this too. But once again, matters are simpler in hybrid languages. It is straightforward to bind nominals classically with $\exists$ and $\forall$.[28] Moreover, recent work on analytic proofs systems for hybrid

[27]This *correspondence-theoretic* perspective is strongly emphasized in the work of Johan van Benthem and other Amsterdam logicians, and traces back to van Benthem [3]. For an up-do-date textbook treatment of modal logic which takes this perspective as its starting point, see Blackburn, de Rijke and Venema [10].

[28]Robert Bull did so in his 1970 paper (see Bull [17] , the Sofia School looked at such languages in the mid 1980s (see Passy and Tinchev [39] and there are a number of recent papers on the topic (for example Blackburn and Tzakova [14] .

languages shows that once the underlying propositional systems have been properly handled (the tricky part is devising decent mechanisms for the modalities) the mechanisms for the binders can be bolted on afterwards in a modular way.[29] Moreover, we don't need to jump all the way up to the full first-order quantification offered by $\forall$ and $\exists$; there are interesting intermediate systems (notably those using the $\downarrow$ binder which binds a nominal to the point of evaluation) and these have also turned out to be logically natural (see Areces, Blackburn and Marx [2]).

In short, hybrid languages offer a framework which takes us slowly up an expressivity hierarchy from simple propositional systems up to rich languages offering full quantification, always treating different types of information in an even-handed way: all in all, a natural setting for developing fine grained theories of time.

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