Relative Similarity Logics are Decidable: Reduction to FO^2 with Equality*

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Abstract. We show the decidability of the satisfiability problem for relative similarity logics that allow classification of objects in presence of incomplete information. As a side-effect, we obtain a finite model property for such similarity logics. The proof technique consists of reductions into the satisfiability problem for the decidable fragment FO^2 with equality from classical logic. Although the reductions stem from the standard translation from modal logic into classical logic, our original approach (for instance handling nominals for atomic properties and decomposition in terms of components encoded in the reduction) can be generalized to a larger class of relative logics, opening ground for further investigations.

1 Introduction

Background. Classification of objects in presence of incomplete information has been long recognized as an issue of concern for various AI problems that deal with commonsense knowledge as well as scientific and engineering knowledge (expert systems, image recognition, knowledge bases and so on). Similarity -sometimes termed "weak equivalence"- provides a basic tool each time when we classify objects with respect to their properties. There exist several formal systems capturing the notion of similarity from the logical viewpoint [Vak91a,Vak91b]. In the present paper we base on the formalization given in [Kon97], where, contrary to [Vak91a,Vak91b], similarity is treated as a relative notion. More precisely, in [Kon97] similarity is defined as a reflexive and symmetric binary relation sim_P, parametrized by the set P of properties with respect to which the objects are classified as either similar or dissimilar. Thus, instead of a single similarity relation we have a whole family $(sim_P)_{P \subseteq PROP}$, where $PROP$ is the set of all the properties considered in a given system. When talking about similarity or equivalence it is natural to talk about *lower* and *upper approximation* $L(sim_P)A$, $U(sim_P)A$ of a given set A of objects with respect to the similarity sim_P. The above operations stem from rough set theory [Paw81], with $L(sim_P)A$ being the set of all objects in A which are not similar (in the sense of sim_P) to any

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object outside A and $U(sim_P)A$ - the set of all objects of the universe which are similar to some object in A. Thus, the above operations could be considered as the operations of taking "interior" and "closure" of the set A with respect to similarity sim_P. However, the analogy is not complete, since similarity is not transitive, and hence the above operations are not idempotent.

Practical importance of the approximation operations is quite obvious: if we can distinguish objects only up to similarity, then when looking for objects belonging to some set A we should take those in $L(sim_P)A$, if we want to consider only the objects sure to belong to A, and those in $U(sim_P)A$ if our aim is not to overlook any object which might possibly belong to A.

Our objectives. The formal system introduced in [Kon97] features the above operations, which generate a family of interdependent relative modalities. The resulting polymodal logic is equipped with a complete deduction system. However, from the viewpoint of any practical applications of the similarity logic in the area of Artificial Intelligence mentioned above an issue of great importance is whether the logic is decidable. A positive answer to this question might provide not only a decision procedure, but also a better understanding of the logical analysis of similarity. These are the objectives of the present paper. Up to now, the question of decidability has been open, which is hardly surprising in view of the high expressive power of the logic. Indeed: its language admits implicitly the universal modal operator, and nominals for atomic propositions as well for atomic properties; in addition, the modal operators are interdependent. Nominals (or names) are used in numerous non-classical logics with various motivations (see e.g. [Orło84a,PT91,Bla93,Kon97]) and they usually greatly increase the expressive power of the logics (causing additional difficulties with proving (un)decidability -see e.g. [PT91]). Furthermore, since finite submodels can be captured in the language up to an isomorphism (which is yet another evidence of the expressive power of similarity logics), there is no hope of proving decidability by showing a finite model property for a class of models including strictly the class of standard models with a bound on the model's size (see e.g. [Vak91c,Bal97]). On the other hand, the intersection operator, which is implicitly present in the interpretation of the modal terms, is known to behave badly for filtration-like constructions.

Our contribution. We prove that the logic defined in [Kon97] together with some of its variants is decidable by translating it to a decidable fragment of first-order logic: the two-variable fragment FO^2 containing equality, but no function symbols (see e.g. [Mor75]). Although there are known methods of handling the universal modal operator, the Boolean operations for modal terms and nominals for atomic propositions in order to translate them into FO^2 with equality (see for example the survey papers [Ben98,Var97]), the extra features of the similarity logics require some significant extra work in order to be also translated to such a fragment. This is achieved in the present paper. Unlike the Boolean Modal Logic BML [GP90], for which decidability can be proved via the finite model property for a class of models, reduction of satisfiability for the similarity logics to FO^2 with equality is the only known decidability proof we are aware of, and therefore we solve an open problem here. As a side-effect, we prove the finite model property. More importantly, the novelty of our approach allows us to generalize the translation to a large class of relative modal logics.

Plan of the paper. The paper is structured as follows. In Section 2 the relative similarity logics we deal with in the paper are defined, and some results about their expressive power and complexity are stated. In Section 3, we define the translation of the main relative similarity logic $\mathcal{L}$ into FO^2 with equality, and show its faithfulness. Decidability and finite model property for $\mathcal{L}$ are obtained partly by considering the analogous properties of the fragment FO^2 with equality. In Section 4, we investigate some variants of $\mathcal{L}$, and show their decidability and the finite model property. Section 5 concludes the paper by providing some generalizations of the results proved in the preceding Sections, and stating what is known about the computational complexity of $\mathcal{L}$-satisfiability. In addition, several examples of formula translations are given.

2 Similarity logics

2.1 Information systems and similarity

The information systems that proposed for representation of knowledge are the foundational structures, on which the semantics of the relative similarity logic is based. An *information system* S is defined as a pair $\langle ENT, PROP \rangle$ where ENT is a non-empty set of *entities* (also called *objects*) and $PROP$ is a non-empty set of *properties* (also called *attributes*) -see e.g. [Paw81]. Each property *prop* is a mapping $ENT \rightarrow \mathcal{P}(Val_{prop}) \setminus \emptyset$ and Val_{prop} is the set of *values* of the property *prop* -see e.g. [OP84]. In that setting, two entities e_1, e_2 are said to be *similar with respect to some set* $P \subseteq PROP$ *of properties* (in short $e_1 \ sim_P \ e_2$) iff for any $prop \in P$, $prop(e_1) \cap prop(e_2) \neq \emptyset$. The polymodal frames of the relative similarity logics are isomorphic to structures of the form $(ENT, PROP, (sim_P)_{P \subseteq PROP})$. Other relationships between entities can be found in the literature -see e.g. [FdCO84,Orło84b]. For instance, two entities e_1, e_2 are said to be *negatively similar (resp. indiscernible) with respect to some set* $P \subseteq PROP$ *of properties* (in short $e_1 \ nsim_P \ e_2$ -resp. $e_1 \ ind_P \ e_2$) iff for any $prop \in P$, $-prop(e_1) \cap -prop(e_2) \neq \emptyset$ - resp. $prop(e_1) = prop(e_2)$.

The family $(sim_P)_{P \subseteq PROP}$ of similarity relations stemming from some information system $S = \langle ENT, PROP \rangle$ induces certain approximations of subsets of entities in S. Indeed, let $L(sim_P)X$ (resp. $U(sim_P)X$) be the lower (resp. upper) sim_P-approximation of the set X of entities defined as follows:

- $L(sim_P)X \stackrel{\text{def}}{=} \{e \in ENT : \forall \ e' \in ENT, \ (e, e') \in sim_P$ implies $e' \in X\}$;
- $U(sim_P)X \stackrel{\text{def}}{=} \{e \in ENT : \exists \ e' \in ENT, \ (e, e') \in sim_P$ and $e' \in X\}$.

Obviously $L(sim_P)X \subseteq X \subseteq U(sim_P)X$ and $L(sim_P)X = ENT \setminus U(sim_P)(ENT \setminus X)$. These approximations are rather crucial in rough set theory since they allow to classify objects in presence of incomplete information. That is why, the semantics of modal operators in the relative similarity logics shall use these approximations as modal operations. We invite the reader to consult [Oe97] for examples of rough set analysis of incomplete information.

2.2 Syntax and semantics

The set of primitive symbols of the polymodal language $\mathtt{L}$ is composed of

- a set $\mathtt{VARSE} = \{\mathtt{E}_1, \mathtt{E}_2, \ldots\}$ of variables representing sets of entities,
- a set $\mathtt{VARE} = \{\mathtt{x}_1, \mathtt{x}_2, \ldots\}$ of variables representing individual entities,
- symbols for the classical connectives $\neg$, $\wedge$ (negation and conjunction), and
- a countably infinite set $\{[\mathtt{A}] : \mathtt{A} \in \mathtt{TERM}\}$ of unary modal operators where the set $\mathtt{TERM}$ of terms is the smallest set containing
  - the constant $\mathtt{0}$ representing the empty set of properties,
  - a countably infinite set $\mathtt{VARP} = \{\mathtt{p}_1, \mathtt{p}_2, \ldots\}$ of variables representing individual properties,
  - a countably infinite set $\mathtt{VARSP} = \{\mathtt{P}_1, \mathtt{P}_2, \ldots\}$ of variables representing sets of properties,

  and closed under the Boolean operators $\cap, \cup, -$.

The formation rules of the set $\mathtt{FORM}$ of formulae are those of the classical propositional calculus plus the rule: if $\mathtt{F} \in \mathtt{FORM}$ and $\mathtt{A} \in \mathtt{TERM}$, then $[\mathtt{A}]\mathtt{F} \in \mathtt{FORM}$. We use the connectives $\vee, \Rightarrow, \Leftrightarrow, \langle\mathtt{A}\rangle$ as abbreviations with their standard meanings. For any syntactic category $\mathtt{X}$ and any syntactic object $\mathtt{O}$, we write $\mathtt{X}(\mathtt{O})$ to denote the set of those elements of $\mathtt{X}$ that occur in $\mathtt{O}$. Moreover, for any syntactic object $\mathtt{O}$, we write $|\mathtt{O}|$ to denote its *length* (or *size*), that is the number of symbol occurrences in $\mathtt{O}$. As usual, $sub(\mathtt{F})$ denotes the set of *subformulae* of the formula $\mathtt{F}$ (including $\mathtt{F}$ itself).

Definition 1. *A* $\mathtt{TERM}$-interpretation *v is a map $v : \mathtt{TERM} \rightarrow \mathcal{P}(PROP)$ such that $PROP$ is a non-empty set and for any $\mathtt{A}_1, \mathtt{A}_2 \in \mathtt{TERM}$,*

- *if $\mathtt{A}_1, \mathtt{A}_2 \in \mathtt{VARP}$ and $\mathtt{A}_1 \neq \mathtt{A}_2$, then $v(\mathtt{A}_1) \neq v(\mathtt{A}_2)$,*
- *if $\mathtt{A}_1 \in \mathtt{VARP}$, then $v(\mathtt{A}_1)$ is a singleton, i.e. $v(\mathtt{A}_1) = \{prop\}$ for some $prop \in PROP$,*
- *$v(\mathtt{0}) = \emptyset$, $v(\mathtt{A}_1 \cap \mathtt{A}_2) = v(\mathtt{A}_1) \cap v(\mathtt{A}_2)$, $v(\mathtt{A}_1 \cup \mathtt{A}_2) = v(\mathtt{A}_1) \cup v(\mathtt{A}_2)$,*
- *$v(-\mathtt{A}_1) = PROP \setminus v(\mathtt{A}_1)$.*

For any $\mathtt{A}, \mathtt{B} \in \mathtt{TERM}$, we write $\mathtt{A} \equiv \mathtt{0}$ (resp. $\mathtt{A} \equiv \mathtt{B}$) when for any $\mathtt{TERM}$-interpretation v, $v(\mathtt{A}) = \emptyset$ (resp. $v(\mathtt{A}) = v(\mathtt{B})$).

Definition 2. *A* model *$\mathcal{U}$ is a structure $\mathcal{U} = (ENT, PROP, (sim_P)_{P \subseteq PROP}, v)$ where ENT and $PROP$ are non-empty sets and $(sim_P)_{P \subseteq PROP}$ is a family of binary relations over ENT such that*

- *for any $\emptyset \neq P \subseteq PROP$, sim_P is reflexive and symmetric,*
- *for any $P, P' \subseteq PROP$, $sim_{P \cup P'} = sim_P \cap sim_{P'}$ and $sim_\emptyset = ENT \times ENT$.*

Moreover, v is a mapping v : $\mathtt{VARE} \cup \mathtt{VARSE} \cup \mathtt{TERM} \rightarrow \mathcal{P}(ENT) \cup \mathcal{P}(PROP)$ such that $v(\mathtt{E}) \subseteq ENT$ for any $\mathtt{E} \in \mathtt{VARSE}$, $v(\mathtt{x}) = \{e\}$, where $e \in ENT$ for any $\mathtt{x} \in \mathtt{VARE}$ and the restriction of v to TERM *is a* TERM*-interpretation.*

Since the set of nominals for properties is countably infinite, and any two different nominals are interpreted by different properties, each model has an infinite set of properties. Let $\mathcal{U} = (ENT, PROP, (sim_P)_{P \subseteq PROP}, v)$ be a model. As usual, we say that a formula F is *satisfied by an entity $e \in ENT$ in $\mathcal{U}$* (written $\mathcal{U}, e \models \mathtt{F}$) if the following conditions are satisfied.

- $\mathcal{U}, e \models \mathtt{x}$ iff $\{e\} = v(\mathtt{x})$; $\mathcal{U}, e \models \mathtt{E}$ iff $e \in v(\mathtt{E})$;
- $\mathcal{U}, e \models \neg\mathtt{F}$ iff not $\mathcal{U}, e \models \mathtt{F}$; $\mathcal{U}, e \models \mathtt{F} \wedge \mathtt{G}$ iff $\mathcal{U}, e \models \mathtt{F}$ and $\mathcal{U}, e \models \mathtt{G}$;
- $\mathcal{U}, e \models [\mathtt{A}]\mathtt{F}$ iff for any $e' \in sim_{v(\mathtt{A})}(e)$, $\mathcal{U}, e' \models \mathtt{F}$.

A formula F is *true* in a model $\mathcal{U}$ (written $\mathcal{U} \models \mathtt{F}$) iff for any $e \in ENT$, $\mathcal{U}, e \models \mathtt{F}$ - or, equivalently, iff for some $e \in ENT$, $\mathcal{U}, e \models [\mathtt{0}]\mathtt{F}$. A formula F is said to be *valid* iff F is true in all models. A formula F is said to be *satisfiable* iff $\neg\mathtt{F}$ is not valid. The similarity logic $\mathcal{L}$ is said to have the *finite model property* iff every satisfiable formula is satisfied in some model $\mathcal{U} = (ENT, PROP, (sim_P)_{P \subseteq PROP}, v)$ with a finite set ENT such that, for any $P \subseteq PROP$, $sim_P = sim_{P \cap P_0}$, where $P_0 \subseteq PROP$ is finite and nonempty (P_0 is called the *relevant part* of $PROP$ in $\mathcal{U}$). Consequently, if $\mathcal{L}$ has the finite model property, then every satisfiable formula has a model $(ENT, PROP, (sim_P)_{P \subseteq PROP}, v)$ such that for any $\emptyset \neq P \subseteq PROP$, $sim_P = \cap_{x \in P} sim_{\{x\}}$.

The similarity logic defined in [Kon97] is not exactly the logic $\mathcal{L}$ defined above, since in [Kon97] the set of properties was supposed to be fixed, and constants representing properties were used instead of variables. For any set X, we write $\mathcal{L}_X$ to denote the logic that differs from $\mathcal{L}$ in the following points: (1) the set of properties $PROP$ is fixed in all the models and equals X, (2) VARP and X have the same cardinality. In various places in the paper, we implicitly use the facts that satisfiability is insensitive to the renaming of any sort of variables, and that any two models isomorphic in the standard sense satisfy the same set of formulae. Moreover, for the logics $\mathcal{L}_X$, as far as satisfiability is concerned, it is irrelevant whether we fix the interpretation of each nominal for the properties.

2.3 Expressive power and complexity lower bound

Since the language of the relative similarity logic $\mathcal{L}$ contains nominals, the universal modal operator and a family of standard modal operators, its expressive power is quite high. In Proposition 1 below, we shall state a counterpart of Corollary 4.17 in [GG93] (see also Theorem 2.8 in [PT91]) saying that finite submodels can be captured in the language up to isomorphism. In Proposition 1 below, we show that for any finite structure $\mathcal{S}$ there is a formula $\mathtt{F}_{\mathcal{S}}$ such that a model $\mathcal{U}$ satisfies $\mathtt{F}_{\mathcal{S}}$ iff $\mathcal{S}$ is a substructure of $\mathcal{U}$ up to isomorphism. Although this shows that the expressive power of the logic is high, it has a very unpleasant consequence: there is no hope of characterizing $\mathcal{L}$-satisfiability by a class of *finite non-standard* models the way it is done in [Vak91c,Bal97]. It means for instance that proving the finite model property of $\mathcal{L}$ by a standard filtration-like technique becomes highly improbable since $\mathcal{L}$ has implicitly the intersection operator in the language.

In Proposition 1 below, the structure $\mathcal{S}$ encodes a finite part of some model. The set $\{1,\ldots,n\}$ should be understood as a finite set of entities, and $\{1,\ldots,l\}$ as a finite set of properties. Moreover, only a finite set $\{\mathtt{E}_1,\ldots,\mathtt{E}_k\}$ of atomic propositions is taken into account. For $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,k\}$, $j \in v'(i)$ is to mean that $\mathtt{E}_j$ is satisfied by i.

Proposition 1. *Let $\mathcal{S} = \langle \{1,\ldots,n\}, \{1,\ldots,l\}, (R(P))_{P\subseteq\{1,\ldots,l\}}, v' \rangle$ be a structure such that each $R(P)$ is a reflexive and symmetric relation, $R(\emptyset)$ is the universal relation, for any $P, P' \subseteq \{1,\ldots,l\}$, $R(P \cup P') = R(P) \cap R(P')$ and v' is a mapping $\{1,\ldots,n\} \to \mathcal{P}(\{1,\ldots,k\})$ for some $k \geq 1$. Then, there is formula $\mathtt{F}_{\mathcal{S}}$ such that for any $\mathcal{L}$-model $\mathcal{U}$, $\mathcal{U} \models \mathtt{F}_{\mathcal{S}}$ iff there is an 1-1 mapping $\Psi_1 : \{1,\ldots,n\} \to ENT$ and an injective mapping $\Psi_2 : \{1,\ldots,l\} \to PROP$ with the following properties*

- *for any $i \in \{1,\ldots,k\}$, $v(\mathtt{E}_i) = \{\Psi_1(s) : i \in v'(s)\}$;*
- *for any $P \subseteq PROP$ such that there is $P' \subseteq \{1,\ldots,l\}$ verifying $\{\Psi_2(i) : i \in P'\} = P$, we have $sim_P = R(P')$.*

Proof. The formula $\mathtt{F}_{\mathcal{S}}$ is the conjunction of the following formulae.

1. $[0](\mathtt{x}_1 \vee \ldots \vee \mathtt{x}_n) \Leftrightarrow [0] \bigwedge_{1\leq i<j\leq n} \neg(\mathtt{x}_i \wedge \mathtt{x}_j)$;
2. $[0](\bigwedge_{i\in\{1,\ldots,n\}}(\mathtt{x}_i \Rightarrow (\bigwedge_{u\in\{1,\ldots,k\}} s_u \mathtt{E}_u)))$ where s_u is the empty string if $u \in v(i)$, otherwise $s_u \stackrel{\text{def}}{=} \neg$;
3. for any $\{i_1,\ldots,i_q\} \subseteq \{1,\ldots,l\}$ and all $i \in \{1,\ldots,n\}$,

$$[0](\mathtt{x}_i \Rightarrow ((\bigwedge_{j\in R_{\{i_1,\ldots,i_q\}}(i)} \langle \mathtt{p}_{i_1} \cup \ldots \cup \mathtt{p}_{i_q} \rangle \mathtt{x}_j) \wedge (\bigwedge_{j\notin R_{\{i_1,\ldots,i_q\}}(i)} \neg\langle \mathtt{p}_{i_1} \cup \ldots \cup \mathtt{p}_{i_q} \rangle \mathtt{x}_j))$$

Before establishing decidability of $\mathcal{L}$-satisfiability, one can provide a lower bound for the complexity of this problem using [Hem96].

Proposition 2. *$\mathcal{L}$-satisfiability is* **EXPTIME***-hard.*

When no nominals for properties and entities are allowed satisfiability can be shown to be in **EXPTIME** [Dem98].

3 Translation from $\mathcal{L}$ into FO^2 with equality

3.1 A known decidable fragment of classical logic

Consistently with the general convention, by FO^2 we mean a fragment of first-order logic (FOL for short) without equality or function symbols using only 2 variables (denoted by $\mathtt{y}_0$ and $\mathtt{y}_1$ in the sequel). We shall translate the similarity logics into a slight extension of FO^2 obtained by augmenting the language with identity. Actually, we shall restrict ourselves to the following vocabulary:

– a countable set $\{\mathtt{P}_i : i \in \omega\} \cup \{\mathtt{Q}_i : i \in \omega\}$ of unary predicate symbols,
– a countable set $\{\mathtt{R}_{i,j} : i, j \in \omega\}$ of binary predicate symbols,
– the symbol $\mathtt{=}$ (interpreted as identity).

In what follows, by a first-order formula we mean a formula belonging to just this fragment of FOL (written $\mathrm{FO}^2[\mathtt{=}]$ in the sequel). As usual, a first-order structure $\mathcal{M}$ (restricted to this fragment) is a pair $\langle D, m\rangle$ such that D is a non-empty set and m is an interpretation function with $m(\mathtt{P}_i) \cup m(\mathtt{Q}_i) \subseteq D$ for $i \in \omega$, $m(\mathtt{R}_{i,j}) \subseteq D \times D$ for $i, j \in \omega$ and $m(\mathtt{=}) \stackrel{\text{def}}{=} \{\langle a, a\rangle : a \in D\}$. As usual, a *valuation* $v_{\mathcal{M}}$ for $\mathcal{M}$ is a mapping $v_{\mathcal{M}} : \{\mathtt{y}_0, \mathtt{y}_1\} \to D$. We write $\mathcal{M}, v_{\mathcal{M}} \models \mathtt{F}$ to denote that $\mathtt{F}$ is satisfied in $\mathcal{M}$ under $v_{\mathcal{M}}$, and omit $v_{\mathcal{M}}$ when $\mathtt{F}$ is closed. It is known that $\mathrm{FO}^2[\mathtt{=}]$ has the finite model property, $\mathrm{FO}^2[\mathtt{=}]$-satisfiability is decidable [Mor75] and **NEXPTIME**-complete [Lew80,GKV97]. Actually, $\mathtt{F}$ is $\mathrm{FO}^2[\mathtt{=}]$-satisfiable iff $\mathtt{F}$ has a model of size $2^{c \times |\mathtt{F}|}$ for some fixed $c > 0$ [GKV97].

3.2 Normal forms

Let $\mathtt{F} \in \mathtt{FORM}$ be such that[3] $\mathtt{VARP}(\mathtt{F}) = \{\mathtt{p}_1, \ldots, \mathtt{p}_l\}$ and $\mathtt{VARSP}(\mathtt{F}) = \{\mathtt{P}_1, \ldots, \mathtt{P}_n\}$. In the rest of this section, we assume that $n \geq 1$ and $l \geq 1$. The degenerate cases make no additional difficulties and they are treated in a separate section. For any integer $k \in \{0, \ldots, 2^n - 1\}$, by $\mathtt{B}_k$ we denote the term

$$\mathtt{A}_1 \cap \ldots \cap \mathtt{A}_n$$

where, for any $s \in \{1, \ldots, n\}$, $\mathtt{A}_s = \mathtt{P}_s$ if $bit_s(k) = 0$, and $\mathtt{A}_s = -\mathtt{P}_s$ otherwise, with $bit_s(k)$ denoting the sth bit in the binary representation of k. For any integer $k \in \{0, \ldots, 2^n - 1\}$, we denote

$$\mathtt{A}_{k,0} \stackrel{\text{def}}{=} \mathtt{B}_k \cap -\mathtt{p}_1 \cap \ldots \cap -\mathtt{p}_l$$

Finally, for any $\langle k, k'\rangle \in \{0, \ldots, 2^n - 1\} \times \{1, \ldots, l\}$, we denote $\mathtt{A}_{k,k'} \stackrel{\text{def}}{=} \mathtt{B}_k \cap \mathtt{p}_{k'}$. For any $\mathtt{TERM}$-interpretation $v : \mathtt{TERM} \to \mathcal{P}(PROP)$, the family

$$\{v(\mathtt{A}_{k,k'}) : \langle k, k'\rangle \in \{0, \ldots, 2^n - 1\} \times \{0, \ldots, l\}\}$$

is a partition of $PROP$. Moreover, for any term $\mathtt{A} \in \mathtt{TERM}(\mathtt{F})$, either $\mathtt{A} \equiv \mathtt{0}$ or there is a unique non-empty set $\{\mathtt{A}_{k_1,k'_1}, \ldots, \mathtt{A}_{k_u,k'_u}\}$ such that $\mathtt{A} \equiv \mathtt{A}_{k_1,k'_1} \cup \ldots \cup \mathtt{A}_{k_u,k'_u}$. The normal form of $\mathtt{A}$, written $N(\mathtt{A})$, is either $\mathtt{0}$ or $\mathtt{A}_{k_1,k'_1} \cup \ldots \cup \mathtt{A}_{k_u,k'_u}$ according to the two cases above. Such a decomposition, introduced in [Kon97], generalizes with nominals the canonical disjunctive normal form for the propositional calculus. $N(\mathtt{A})$ can be computed by an effective procedure.

For any $k' \in \{1, \ldots, l\}$, we write $occ_{k'}$ to denote the set

$$\{k \in \{0, \ldots, 2^n - 1\} : \exists \mathtt{A} \in \mathtt{TERM}(\mathtt{F}),\ N(\mathtt{A}) = \ldots \cup \mathtt{A}_{k,k'} \cup \ldots\}$$

[3] Without any loss of generality we can assume that if l (resp. n) nominals for properties (resp. for entities) occur in $\mathtt{F}$ they are precisely the l (resp. n) first in the enumeration of $\mathtt{VARP}$ (resp. $\mathtt{VARE}$) since satisfiability is not sensitive to the renaming of variables.

Informally, $occ_{k'}$ is the set of indices k such that $\mathtt{A}_{k,k'}$ occurs in the *normal form* of some element of $\mathtt{TERM}(\mathtt{F})$. We write $setocc_{k'}$ to denote the set

$$\{X \subseteq occ_{k'} : card(occ_{k'}) - 1 \leq card(X) \leq 2^n - 1\}$$

The definition of $setocc_{k'}$ is motivated by the fact that for any TERM-interpretation v, there is only one $k \in \{0, \ldots, 2^n - 1\}$ such that $v(\mathtt{A}_{k,k'}) \neq \emptyset$, and for this very k, $v(\mathtt{A}_{k,k'}) = v(\mathtt{p}_{k'})$. For each $X \in setocc_{k'}$ in turn, in the forthcoming constructions we shall enforce $v(\mathtt{A}_{k,k'}) = \emptyset$ for any $k \in X$.

3.3 The translation

In this section, we define an extension to $\mathcal{L}$ of the translation ST defined in [Ben83] of modal formulae into a first-order language containing a binary predicate, a countable set of unary predicate symbols and two individual variables (due to a smart recycling of the variables). Our translation of the nominals for entities is similar to the translation of nominals in [GG93]. However, we take into account the decomposition of terms into components in order to obtain a faithful translation. The translation of nominals for atomic properties is a twofold one: we take it into account both in defining the normal form of terms, and in the generalized disjunction defining the translation T below.

Let $\mathtt{F} \in \mathtt{FORM}$ be such that $\mathtt{VARP}(\mathtt{F}) = \{\mathtt{p}_1, \ldots, \mathtt{p}_l\}$, $\mathtt{VARSP}(\mathtt{F}) = \{\mathtt{P}_1, \ldots, \mathtt{P}_n\}$ and $\mathtt{VARE}(\mathtt{F}) = \{\mathtt{x}_1, \ldots, \mathtt{x}_q\}$. Before defining ST' - the mapping translating $\mathcal{L}$-formulae into FO^2-formulae - let us state what are the main features we intend that mapping to have. Analogously to ST, ST' encodes the quantification in the interpretation of $[\mathtt{A}]$ into the language of FO^2 by using the standard universal quantifier $\forall$ and by introducing a binary predicate symbol $\mathtt{R}_{\mathtt{A}}$ for each $\mathtt{A} \in \mathtt{TERM}$. However, this is not exactly the way ST' is defined. Actually, to each component $\mathtt{A}_{k,k'}$ we associate the predicate symbol $\mathtt{R}_{k,k'}$. The main idea of ST' is therefore to treat components as constants, which means that the translation of $[\mathtt{A}]\mathtt{G}$ is uniquely determined by the components (if any) of the normal form of A. Then, the conditions on the $\mathcal{L}$-models justify why a modal operator indexed by the *union* of components is translated into a formula involving a *conjunction* of atomic formulae. Let ST' be defined as follows (ST' is actually parametrized by F and $i \in \{0, 1\}$):

(1) $ST'(\mathtt{E}_j, \mathtt{y}_i) \stackrel{\text{def}}{=} \mathtt{P}_j(\mathtt{y}_i)$; $ST'(\mathtt{x}_j, \mathtt{y}_i) \stackrel{\text{def}}{=} \mathtt{Q}_j(\mathtt{y}_i)$;
(2) $ST'(\neg\mathtt{G}, \mathtt{y}_i) \stackrel{\text{def}}{=} \neg ST'(\mathtt{G}, \mathtt{y}_i)$; $ST'(\mathtt{G} \wedge \mathtt{H}, \mathtt{y}_i) \stackrel{\text{def}}{=} ST'(\mathtt{G}, \mathtt{y}_i) \wedge ST'(\mathtt{H}, \mathtt{y}_i)$;
(3)

$$ST'([\mathtt{A}]\mathtt{G}, \mathtt{y}_i) \stackrel{\text{def}}{=} \begin{cases} \forall\ \mathtt{y}_0\ ST'(\mathtt{G}, \mathtt{y}_0) \text{ if } N(\mathtt{A}) = \mathtt{0} \\ \forall\ \mathtt{y}_{1-i}(\mathtt{R}_{k_1,k'_1}(\mathtt{y}_i, \mathtt{y}_{1-i}) \wedge \ldots \wedge \mathtt{R}_{k_u,k'_u}(\mathtt{y}_i, \mathtt{y}_{1-i})) \Rightarrow ST'(\mathtt{G}, \mathtt{y}_{1-i}) \\ \text{if } N(\mathtt{A}) = \mathtt{A}_{k_1,k'_1} \cup \ldots \cup \mathtt{A}_{k_u,k'_u} \end{cases}$$

By adopting the standard definition $\langle\mathtt{A}\rangle\mathtt{G} \stackrel{\text{def}}{=} \neg[\mathtt{A}]\neg\mathtt{G}$, ST' can be easily defined for $\langle\mathtt{A}\rangle\mathtt{G}$: the existential quantification is involved instead of universal one.

Let $\mathtt{G}_0$ be a first-order formula (in FO^2) expressing the fact that, for any $\langle k, k'\rangle \in \{0, \ldots, 2^n - 1\} \times \{0, \ldots, l\}$, $\mathtt{R}_{k,k'}$, is interpreted as a reflexive and symmetric binary relation. Let $\mathtt{G}_1$ be a first-order formula expressing the fact that, for any $i \in \{1, \ldots, q\}$, $\mathtt{Q}_i$ is interpreted[4] as a singleton, e.g.

$$\bigwedge_{i=1}^{q} \exists\ \mathtt{y}_0\ (\mathtt{Q}_i(\mathtt{y}_0) \wedge \forall\ \mathtt{y}_1\ \neg \mathtt{y}_0 = \mathtt{y}_1 \Rightarrow \neg \mathtt{Q}_i(\mathtt{y}_1))$$

In the case when $\mathtt{VARE}(\mathtt{F}) = \emptyset$, $\mathtt{G}_1 \stackrel{\text{def}}{=} \forall \mathtt{y}_0\ \mathtt{y}_0 = \mathtt{y}_0$. Let $\mathtt{T}_1(\mathtt{F})$ be the first-order formula (in $\mathrm{FO}^2[\mathtt{=}]$) defined by

$$\mathtt{T}_1(\mathtt{F}) \stackrel{\text{def}}{=} \mathtt{G}_0 \wedge \mathtt{G}_1 \wedge \exists\ \mathtt{y}_0\ ST'(\mathtt{F}, \mathtt{y}_0)$$

The translation is not quite finished yet. Indeed, although at least one of the components $\mathtt{p}_1 \cap \mathtt{P}_1$ or $\mathtt{p}_1 \cap -\mathtt{P}_1$ is interpreted by the empty set of properties, this fact is not taken into account in ST' (considering e.g. $n = 1$). This is a serious gap since at least one of the predicate symbols $\mathtt{R}_{0,1}$ or $\mathtt{R}_{1,1}$ should be interpreted as the universal relation. The forthcoming developments provide an answer to this technical problem.

Let $\mathtt{G}$ be a first-order formula, $k' \in \{1, \ldots, l\}$ and $X_{k'} \in setocc_{k'}$. We write $\mathtt{G}[k', X_{k'}]$ to denote the first-order formula obtained from $\mathtt{G}$ by substituting:

- every occurrence of $\mathtt{R}_{k,k'}(\mathtt{z}_1, \mathtt{z}_2) \Rightarrow \mathtt{H}$ with $\mathtt{H}$ if $k \in X_{k'}$,
- every occurrence of $\mathtt{F}' \wedge \mathtt{R}_{k,k'}(\mathtt{z}_1, \mathtt{z}_2) \wedge \mathtt{F}'' \Rightarrow \mathtt{H}$ with $\mathtt{F}' \wedge \mathtt{F}'' \Rightarrow \mathtt{H}$ if $k \in X_{k'}$ -the degenerate cases are omitted here-

(this rewriting procedure is confluent and always terminates). From a semantical viewpoint, the substitution is equivalent[5] to satisfaction of the condition ($k \in X_{k'}$): $\forall \mathtt{z}_1, \mathtt{z}_2,\ \mathtt{R}_{k,k'}(\mathtt{z}_1, \mathtt{z}_2)$. For $\langle X_1, \ldots, X_l\rangle \in setocc_1 \times \ldots \times setocc_l$, we write $\mathtt{G}[X_1, \ldots, X_l]$ to denote the first-order formula $\mathtt{G}[1, X_1][2, X_2] \ldots [l, X_l]$. Observe that for any permutation σ on $\{1, \ldots, l\}$,

$$\mathtt{G}[\sigma(1), X_{\sigma(1)}][\sigma(2), X_{\sigma(2)}] \ldots [\sigma(l), X_{\sigma(l)}] = \mathtt{G}[1, X_1][2, X_2] \ldots [l, X_l]$$

Let $\mathtt{T}(\mathtt{F})$ be the formula

$$\mathtt{T}(\mathtt{F}) \stackrel{\text{def}}{=} \bigvee \{\mathtt{T}_1(\mathtt{F})[X_1, \ldots, X_l] : \langle X_1, \ldots, X_l\rangle \in setocc_1 \times \ldots \times setocc_l\}$$

Observe that $\mathtt{T}$ is exponential-time in $|\mathtt{F}|$ and the size of the formula obtained by translation may increase exponentially. It is however, not clear whether there

[4] Let $\mathrm{FO}^2[\exists^{=1}]$ be FO^2 augmented with the existential quantifier $\exists^{=1}$ meaning "there exists exactly one". $\mathrm{FO}^2[\exists^{=1}]$-satisfiability has been proved to be in **NEXPTIME** (see e.g. [PST97]). By defining $\mathtt{G}_1$ by $\mathtt{G}_1 \stackrel{\text{def}}{=} \bigwedge_{i=1}^{q} \exists^{=1}\ \mathtt{y}_0\ \mathtt{Q}_i(\mathtt{y}_0)$ we are able to prove decidability of $\mathcal{L}$-satisfiability via a translation into $\mathrm{FO}^2[\exists^{=1}]$.

[5] Another solution consists in defining $\mathtt{G}[k', X_{k'}]$ as the formula $(\bigwedge_{k \in X_{k'}} \forall \mathtt{y}_0, \mathtt{y}_1,\ \mathtt{R}_{k,k'}(\mathtt{y}_0, \mathtt{y}_1)) \wedge \mathtt{G}$.

exists a tighter translation that characterizes more accurately the complexity class of $\mathcal{L}$-satisfiability. Observe also that $\mathtt{T}(\mathtt{F})$ is classically equivalent to

$$\mathtt{G}_0 \wedge \mathtt{G}_1 \wedge \exists \mathtt{y}_0 \bigvee \{ST'(\mathtt{F}, \mathtt{y}_0)[X_1, \ldots, X_l] : \langle X_1, \ldots, X_l \rangle \in setocc_1 \times \ldots \times setocc_l\}$$

Example 1. Let $\mathtt{F}$ be the formula $\langle \mathtt{p}_1 \cup \mathtt{p}_2 \rangle \neg \mathtt{E}_1 \wedge [\mathtt{P}_1]\mathtt{E}_1$. Then $\mathtt{F}$ is $\mathcal{L}$-satisfiable, and the translation of $\mathtt{F}$ is the disjunction of the following formulae:

1. $\mathtt{G}_0 \wedge \mathtt{G}_1 \wedge \exists \mathtt{y}_0\ (\exists \mathtt{y}_1\ \mathtt{R}_{1,1}(\mathtt{y}_0, \mathtt{y}_1) \wedge \mathtt{R}_{1,2}(\mathtt{y}_0, \mathtt{y}_1) \wedge \neg \mathtt{P}_1(\mathtt{y}_1)) \wedge (\forall \mathtt{y}_1\ \mathtt{R}_{0,0}(\mathtt{y}_0, \mathtt{y}_1) \Rightarrow \mathtt{P}_1(\mathtt{y}_1))$
2. $\mathtt{G}_0 \wedge \mathtt{G}_1 \wedge \exists \mathtt{y}_0\ (\exists \mathtt{y}_1\ \mathtt{R}_{0,1}(\mathtt{y}_0, \mathtt{y}_1) \wedge \mathtt{R}_{1,2}(\mathtt{y}_0, \mathtt{y}_1) \wedge \neg \mathtt{P}_1(\mathtt{y}_1)) \wedge (\forall \mathtt{y}_1\ \mathtt{R}_{0,0}(\mathtt{y}_0, \mathtt{y}_1) \wedge \mathtt{R}_{0,1}(\mathtt{y}_0, \mathtt{y}_1) \Rightarrow \mathtt{P}_1(\mathtt{y}_1))$
3. $\mathtt{G}_0 \wedge \mathtt{G}_1 \wedge \exists \mathtt{y}_0\ (\exists \mathtt{y}_1\ \mathtt{R}_{1,1}(\mathtt{y}_0, \mathtt{y}_1) \wedge \mathtt{R}_{0,2}(\mathtt{y}_0, \mathtt{y}_1) \wedge \neg \mathtt{P}_1(\mathtt{y}_1)) \wedge (\forall \mathtt{y}_1\ \mathtt{R}_{0,0}(\mathtt{y}_0, \mathtt{y}_1) \wedge \mathtt{R}_{0,2}(\mathtt{y}_0, \mathtt{y}_1) \Rightarrow \mathtt{P}_1(\mathtt{y}_1))$
4. $\mathtt{G}_0 \wedge \mathtt{G}_1 \wedge \exists \mathtt{y}_0\ (\exists \mathtt{y}_1\ \mathtt{R}_{0,1}(\mathtt{y}_0, \mathtt{y}_1) \wedge \mathtt{R}_{0,2}(\mathtt{y}_0, \mathtt{y}_1) \wedge \neg \mathtt{P}_1(\mathtt{y}_1)) \wedge (\forall \mathtt{y}_1\ \mathtt{R}_{0,0}(\mathtt{y}_0, \mathtt{y}_1) \wedge \mathtt{R}_{0,1}(\mathtt{y}_0, \mathtt{y}_1) \wedge \mathtt{R}_{0,2}(\mathtt{y}_0, \mathtt{y}_1) \Rightarrow \mathtt{P}_1(\mathtt{y}_1))$

The translation takes into account that $N(\mathtt{p}_1 \cup \mathtt{p}_2) = \mathtt{A}_{0,1} \cup \mathtt{A}_{1,1} \cup \mathtt{A}_{0,2} \cup \mathtt{A}_{1,2}$ and $N(\mathtt{P}_1) = \mathtt{A}_{0,0} \cup \mathtt{A}_{0,1} \cup \mathtt{A}_{0,2}$.

3.4 Faithfulness of the translation

The rest of this section is devoted to proving Proposition 3 below and stating certain corollaries (some being consequences of the proof of Proposition 3).

Proposition 3. *(1)* $\mathtt{F}$ *is* $\mathcal{L}$*-satisfiable iff (2)* $\mathtt{T}(\mathtt{F})$ *is first-order satisfiable.*

Proof. (1) implies (2). First assume $\mathcal{U}, e_0 \models \mathtt{F}$ for some model

$$\mathcal{U} = (ENT, PROP, (sim_P)_{P \subseteq PROP}, v)$$

and $e_0 \in ENT$ (this is the easier part of the proof). Let us define the following first-order structure $\mathcal{M} \stackrel{\text{def}}{=} \langle D, m \rangle$:

- $D \stackrel{\text{def}}{=} ENT$; for any $i \in \omega$, $m(\mathtt{Q}_i) \stackrel{\text{def}}{=} v(\mathtt{x}_i)$ and $m(\mathtt{P}_i) \stackrel{\text{def}}{=} v(\mathtt{E}_i)$;
- for any $\langle k, k' \rangle \in \{0, \ldots, 2^n - 1\} \times \{1, \ldots, l\}$, $m(\mathtt{R}_{k,k'}) \stackrel{\text{def}}{=} sim_{v(\mathtt{A}_{k,k'})}$ (for the other values of $\langle k, k' \rangle$ the interpretation of $\mathtt{R}_{k,k'}$ is not constrained).

Let $\langle i_1, \ldots, i_l \rangle \in \{0, \ldots, 2^n - 1\}^l$ be such that for any $k' \in \{1, \ldots, l\}$, $v(\mathtt{A}_{i_{k'}, k'}) = v(\mathtt{p}_{k'})$. Such a sequence $\langle i_1, \ldots, i_l \rangle$ is unique. So, for any $k' \in \{1, \ldots, l\}$, $X_{k'} \stackrel{\text{def}}{=} occ_{k'} \setminus \{i_{k'}\}$. It is easy to show that $\mathcal{M} \models \mathtt{G}_0 \wedge \mathtt{G}_1$ since $\mathcal{U}$ is a model. We claim that $\mathcal{M} \models \mathtt{T}_1(\mathtt{F})[X_1, \ldots, X_l]$, and therefore $\mathcal{M} \models \mathtt{T}(\mathtt{F})$. To prove such a result, let us show that for any $\mathtt{G} \in sub(\mathtt{F})$, $e \in ENT$, $i \in \{0, 1\}$, $\mathcal{U}, e \models \mathtt{G}$ iff $\mathcal{M}, v_{\mathcal{M}}[\mathtt{y}_i \leftarrow e] \models ST'(\mathtt{G}, \mathtt{y}_i)[X_1, \ldots, X_l]$. We write $v_{\mathcal{M}}[\mathtt{y}_i \leftarrow e]$ to denote a first-order valuation $v_{\mathcal{M}}$ such that $v_{\mathcal{M}}(\mathtt{y}_i) = e$. It entails $\mathcal{M} \models \exists \mathtt{y}_0\ ST'(\mathtt{G}, \mathtt{y}_0)[X_1, \ldots, X_l]$. We omit the base case and the cases in the induction step when the outermost connective is Boolean. Here are the remaining cases.
Case 1: $\mathtt{G} = [\mathtt{A}]\mathtt{F}_1$ and $N(\mathtt{A}) = \mathtt{0}$

$\mathcal{U}, e \models [\mathtt{A}]\mathtt{F}_1$ iff for any $e' \in sim_{v(\mathtt{A})}(e)$, $\mathcal{U}, e' \models \mathtt{F}_1$
iff for any $e' \in ENT$, $\mathcal{U}, e' \models \mathtt{F}_1$
iff for any $e' \in D$, $\mathcal{M},\ v_{\mathcal{M}}[\mathtt{y}_i \leftarrow e'] \models ST'(\mathtt{F}_1, \mathtt{y}_i)[X_1, \ldots, X_l]$
iff $\mathcal{M} \models \forall \mathtt{y}_0\ ST'(\mathtt{F}_1, \mathtt{y}_0)[X_1, \ldots, X_l]$
iff $\mathcal{M} \models ST'([\mathtt{A}]\mathtt{F}_1, \mathtt{y}_i)[X_1, \ldots, X_l]$

Case 2: $\mathtt{G} = [\mathtt{A}]\mathtt{F}_1$ and $N(\mathtt{A}) = \mathtt{A}_{k_1,k'_1} \cup \ldots \cup \mathtt{A}_{k_u,k'_u}$
First observe that for any $k' \in \{1, \ldots, l\}$ and $k \in X_{k'}$, $v(\mathtt{A}_{k,k'}) = \emptyset$.

$\mathcal{U}, e \models [\mathtt{A}]\mathtt{F}_1$ iff for any $e' \in \bigcap_{i \in \{1,\ldots,u\}} sim_{v(\mathtt{A}_{k_i,k'_i})}(e)$, $\mathcal{U}, e' \models \mathtt{F}_1$
iff for any $e' \in \bigcap_{i \in \{1,\ldots,u\}} sim_{v(\mathtt{A}_{k_i,k'_i})}(e)$,
$\mathcal{M},\ v_{\mathcal{M}}[\mathtt{y}_{1-i} \leftarrow e'] \models ST'(\mathtt{F}_1, \mathtt{y}_{1-i})[X_1, \ldots, X_l]$
iff for any $e' \in \bigcap_{i \in \{1,\ldots,u\}} m(\mathtt{R}_{k_i,k'_i})(e)$,
$\mathcal{M},\ v_{\mathcal{M}}[\mathtt{y}_{1-i} \leftarrow e'] \models ST'(\mathtt{F}_1, \mathtt{y}_{1-i})[X_1, \ldots, X_l]$
iff $\mathcal{M},\ v_{\mathcal{M}}[\mathtt{y}_i \leftarrow e] \models \forall \mathtt{y}_{1-i}\ (\mathtt{R}_{k_1,k'_1}(\mathtt{y}_i, \mathtt{y}_{1-i}) \wedge \ldots \wedge \mathtt{R}_{k_u,k'_u}(\mathtt{y}_i, \mathtt{y}_{1-i})) \Rightarrow$
$ST'(\mathtt{F}_1, \mathtt{y}_{1-i})[X_1, \ldots, X_l]$
iff $\mathcal{M},\ v_{\mathcal{M}}[\mathtt{y}_i \leftarrow e] \models \forall \mathtt{y}_{1-i}\ (\bigwedge_{1 \leq i \leq u,\ k_i \not\in X_{k'_i}} \mathtt{R}_{k_i,k'_i}(\mathtt{y}_i, \mathtt{y}_{1-i})) \Rightarrow$
$ST'(\mathtt{F}_1, \mathtt{y}_{1-i})[X_1, \ldots, X_l]$
($\bigwedge_{1 \leq i \leq u,\ k_i \not\in X_{k'_i}} \mathtt{R}_{k_i,k'_i}(\mathtt{y}_i, \mathtt{y}_{1-i})$ is $\top$ if the conjunction is empty)
iff $\mathcal{M},\ v_{\mathcal{M}}[\mathtt{y}_i \leftarrow e]\ \models (\forall \mathtt{y}_{1-i}\ (\mathtt{R}_{k_1,k'_1}(\mathtt{y}_i, \mathtt{y}_{1-i}) \wedge \ldots \wedge \mathtt{R}_{k_u,k'_u}(\mathtt{y}_i, \mathtt{y}_{1-i})) \Rightarrow$
$ST'(\mathtt{F}_1, \mathtt{y}_{1-i}))[X_1, \ldots, X_l]$
iff $\mathcal{M} \models ST'([\mathtt{A}]\mathtt{F}_1, \mathtt{y}_i)[X_1, \ldots, X_l]$

In the previous line the substitution operation is performed only on $ST'(\mathtt{F}_1, \mathtt{y}_{1-i})$ whereas in the next line it is performed on the whole expression.

(2) implies (1). Omitted because of lack of space.

Corollary 1. *(1) The $\mathcal{L}$-satisfiability problem is decidable. (2) $\mathcal{L}$ has the finite model property. In particular, every $\mathcal{L}$-satisfiable formula* $\mathtt{F}$ *has a model such that* $card(ENT) \leq 2^{2^{p(|\mathtt{F}|)}}$ *for some fixed polynomial $p(n)$, and the cardinality of the relevant part of $\mathcal{U}$ is at most $2^n + l$, where $n = card(\mathtt{VARSP(F)})$ and $l = card(\mathtt{VARP(F)})$.*

As observed by one referee, formalizing concepts from similarity theory directly in first-order logic could be another alternative.

3.5 The degenerate cases

In the previous section we have assumed that $n \geq 1$ and $l \geq 1$. Now let us examine the remaining cases. If $l \geq 1$ and no variable for sets of properties occurs in the formula ($n = 0$), then we consider the following components: $\mathtt{A}_0 = -\mathtt{p}_1 \cap \ldots \cap -\mathtt{p}_l$ and $\mathtt{A}_{k'} = \mathtt{p}_{k'}$ for $k' \in \{1, \ldots, l\}$. Condition (3) in the definition of ST' becomes:

$$ST'([\mathtt{A}]\mathtt{G}, \mathtt{y}_i) \stackrel{\text{def}}{=} \begin{cases} \forall\ \mathtt{y}_0\ ST'(\mathtt{G}, \mathtt{y}_0) \text{ if } N(\mathtt{A}) = 0 \\ \forall\ \mathtt{y}_{1-i}(\mathtt{R}_{0,k'_1}(\mathtt{y}_i, \mathtt{y}_{1-i}) \wedge \ldots \wedge \mathtt{R}_{0,k'_u}(\mathtt{y}_i, \mathtt{y}_{1-i})) \Rightarrow ST'(\mathtt{G}, \mathtt{y}_{1-i}) \\ \text{if } N(\mathtt{A}) = \mathtt{A}_{k'_1} \cup \ldots \cup \mathtt{A}_{k'_u} \end{cases}$$

If $n \geq 1$ and no variable for individual properties occurs in the formula ($l = 0$), then Condition (3) in the definition of ST' becomes:

$$ST'([\mathtt{A}]\mathtt{G}, \mathtt{y}_i) \stackrel{\text{def}}{=} \begin{cases} \forall\ \mathtt{y}_0\ ST'(\mathtt{G}, \mathtt{y}_0) \text{ if } N(\mathtt{A}) = \mathtt{0} \\ \forall\ \mathtt{y}_{1-i}(\mathtt{R}_{k_1,0}(\mathtt{y}_i, \mathtt{y}_{1-i}) \wedge \ldots \wedge \mathtt{R}_{k_u,0}(\mathtt{y}_i, \mathtt{y}_{1-i})) \Rightarrow ST'(\mathtt{G}, \mathtt{y}_{1-i}) \\ \text{if } N(\mathtt{A}) = \mathtt{B}_{k_1} \cup \ldots \cup \mathtt{B}_{k_u} \text{ (see Section 3.2)} \end{cases}$$

Moreover, $\mathtt{T(F)}$ is simply defined as $\mathtt{T}_1\mathtt{(F)}$. In the case when $n = 0$, $l = 0$ and $\mathtt{TERM(F)} \neq \emptyset$, by substituting every occurrence of $\mathtt{0}$ in $\mathtt{F}$ by $\mathtt{p}_1 \cap -\mathtt{p}_1$ we preserve $\mathcal{L}$-satisfiability and reduce the case to the previous one. Otherwise, $\mathtt{F}$ is a formula of the propositional calculus and therefore it poses no difficulty with respect to decidability.

4 Decidability results for variants of $\mathcal{L}$

4.1 Fixed finite set of properties

In this section we consider a finite set $PROP$ of properties, and show that $\mathcal{L}_{PROP}$ shares various features with $\mathcal{L}$. Actually $\mathcal{L}_{PROP}$ corresponds to the similarity logic with a fixed finite set of properties defined in [Kon97]. Without any loss of generality we can assume that $PROP = \{1, \ldots, \alpha\}$ for some $\alpha \geq 1$ and $\mathtt{VARP} = \{\mathtt{p}_1, \ldots, \mathtt{p}_\alpha\}$.

Let $\mathtt{F}$ be a $\mathcal{L}_{PROP}$-formula such that $\mathtt{VARP(F)} = \{\mathtt{p}_{i_1}, \ldots, \mathtt{p}_{i_l}\}$ and $\mathtt{VARSP(F)} = \{\mathtt{P}_{j_1}, \ldots, \mathtt{P}_{j_n}\}$. For any interpretation v possibly occurring in some $\mathcal{L}_{PROP}$-model and for any $\mathtt{A} \in \mathtt{TERM(F)}$, if $v(\mathtt{A}) = \{k_1, \ldots, k_s\}$, then $N_v(\mathtt{A}) \stackrel{\text{def}}{=} \mathtt{p}_{k_1} \cup \ldots \cup \mathtt{p}_{k_s}$ otherwise ($v(\mathtt{A}) = \emptyset$) $N_v(\mathtt{A}) \stackrel{\text{def}}{=} \mathtt{0}$. We write $v_\mathtt{F}$ (resp. $N_v(\mathtt{F})$) to denote the restriction of v to $\mathtt{TERM(F)}$ (resp. the formula obtained from $\mathtt{F}$ by substituting every occurrence of $\mathtt{A}$ by $N_v(\mathtt{A})$). Let $X_\mathtt{F}$ be the *finite* set

$$\{v_\mathtt{F} : v \text{ interpretation possibly occurring in some } \mathcal{L}_{PROP} \text{ -model}\}$$

Proposition 4. *Let* $\mathtt{F}$ *be a* $\mathcal{L}_{PROP}$*-formula. (1)* $\mathtt{F}$ *is* $\mathcal{L}_{PROP}$*-satisfiable iff (2)* $\bigvee_{v'' \in X_\mathtt{F}} N_{v''}(\mathtt{F})$ *is* $\mathcal{L}$*-satisfiable.*

Proof. (1) implies (2): Assume $\mathcal{U} = (ENT, PROP, (sim_P)_{P \subseteq PROP}, v), e_0 \models \mathtt{F}$ for some $e_0 \in ENT$. It is easy to check that $\mathcal{U}', e_0 \models N_v(\mathtt{F})$ where $\mathcal{U}'$ is defined from $\mathcal{U}$ by only replacing v by v' defined as follows: for any $i \in \{1, \ldots, \alpha\}$, $v'(\mathtt{p}_i) \stackrel{\text{def}}{=} \{i\}$. Let $\mathcal{U}'' = (ENT, \omega, (sim''_P)_{P \subseteq \omega}, v'')$ be an $\mathcal{L}$-model such that

- v' and v'' are identical for the common sublanguage,
- for any $i > \alpha$, $v''(\mathtt{p}_i) \stackrel{\text{def}}{=} \{i\}$, for any $P \subseteq \omega$, $sim''_P \stackrel{\text{def}}{=} sim_{P \cap PROP}$.

It is a routine task to check that $\mathcal{U}'', e_0 \models N_v(\mathtt{F})$ and therefore $\mathcal{U}'', e_0 \models \bigvee_{v'' \in X_\mathtt{F}} N_{v''}(\mathtt{F})$. Indeed, for any $\mathtt{A} \in \mathtt{TERM(F)}$, $sim_{v(\mathtt{A})} = sim''_{v''(N_v(\mathtt{A}))}$.

(2) implies (1): Now assume $\bigvee_{v'' \in X_\mathtt{F}} N_{v''}(\mathtt{F})$ is $\mathcal{L}$-satisfiable. There exist an $\mathcal{L}$-model $\mathcal{U}' = (ENT', PROP', (sim'_P)_{P \subseteq PROP'}, v')$, $e_0 \in ENT'$ and $v''_0 \in X_\mathtt{F}$

such that $\mathcal{U}', e_0 \models N_{v_0''}(\mathtt{F})$. By the proof of Proposition 3, we can assume that $\{u_1, \ldots, u_\alpha\}$ is a relevant part of $PROP'$ in $\mathcal{U}'$ such that for any $i \in \{1, \ldots, \alpha\}$, $v'(\mathtt{p}_i) = u_i$. Indeed, $PROP'$ is at least countable, $\mathtt{VARSP}(\bigvee_{v'' \in X_{\mathtt{F}}} N_{v''}(\mathtt{F})) = \emptyset$ and $card(\mathtt{VARP}(\bigvee_{v'' \in X_{\mathtt{F}}} N_{v''}(\mathtt{F}))) \leq \alpha$. Let $\mathcal{U} = (ENT', PROP, (sim_P)_{P \subseteq PROP}, v)$ be the $\mathcal{L}_{PROP}$-model such that:

- v and v_0'' are identical for the common sublanguage;
- for any $P \subseteq PROP$, $sim_P \stackrel{\text{def}}{=} sim'_{\{u_i : i \in P\}}$.

It is a routine task to check that $\mathcal{U}, e_0 \models \mathtt{F}$ since for any $\mathtt{A} \in \mathtt{TERM}(\mathtt{F})$, $sim_{v(\mathtt{A})} = sim'_{v'(N_{v_0''}(\mathtt{A}))}$.

Corollary 2. *$\mathcal{L}_{PROP}$-satisfiability is decidable and $\mathcal{L}_{PROP}$ has the finite model property.*

Example 2. (Example 1 continued) Let $\mathtt{F}$ be the formula $\langle \mathtt{p}_1 \cup \mathtt{p}_2 \rangle \neg \mathtt{E}_1 \wedge [\mathtt{P}_1]\mathtt{E}_1$ for the logic $\mathcal{L}_{\{1,2\}}$. Then, $\mathtt{F}$ *is not* $\mathcal{L}_{\{1,2\}}$-satisfiable, although $\mathtt{F}$ *is* $\mathcal{L}$-satisfiable. The formula $\bigvee_{v' \in X_{\mathtt{F}}} N_{v'}(\mathtt{F})$ is the disjunction of the following formulae:

1. $(\langle \mathtt{p}_1 \cup \mathtt{p}_2 \rangle \neg \mathtt{E}_1 \wedge [\mathtt{0}]\mathtt{E}_1) \vee (\langle \mathtt{p}_1 \cup \mathtt{p}_2 \rangle \neg \mathtt{E}_1 \wedge [\mathtt{p}_1]\mathtt{E}_1)$
2. $(\langle \mathtt{p}_1 \cup \mathtt{p}_2 \rangle \neg \mathtt{E}_1 \wedge [\mathtt{p}_2]\mathtt{E}_1) \vee (\langle \mathtt{p}_1 \cup \mathtt{p}_2 \rangle \neg \mathtt{E}_1 \wedge [\mathtt{p}_1 \cup \mathtt{p}_2]\mathtt{E}_1)$

4.2 Fixed infinite set of properties

In this section we consider some infinite set $PROP$ of properties, and show that $\mathcal{L}_{PROP}$ shares various features with $\mathcal{L}$. Actually, $\mathcal{L}_{PROP}$ corresponds to the similarity logic with a fixed infinite set of properties defined in [Kon97]. Without any loss of generality we can assume that $\omega \subseteq PROP$ (there is an injective map f from ω into $PROP$) and $\{\mathtt{p}_1, \mathtt{p}_2, \ldots\} \subseteq \mathtt{VARP}$.

Proposition 5. *Let $\mathtt{F}$ be a $\mathcal{L}_{PROP}$-formula. (1) $\mathtt{F}$ is $\mathcal{L}_{PROP}$-satisfiable iff (2) $\mathtt{F}$ is $\mathcal{L}$-satisfiable.*

Proof. Omitted because of lack of space.

Corollary 3. *$\mathcal{L}_{PROP}$-satisfiability is decidable and $\mathcal{L}_{PROP}$ has the finite model property.*

5 Concluding remarks

We have shown that the relative similarity logics $\mathcal{L}$ and $\mathcal{L}_X$ for some non-empty set X of properties have a *decidable* satisfiability problems. Moreover, we have also established that such logics have the finite model property. The decidability proof reduces satisfiability in our logic to satisfiability in $\mathrm{FO}^2[=]$, a decidable fragment of classical logic [Mor75]. Although our reduction takes advantage of the standard translation ST [Ben83] of modal logic into classical logic, the novelty of our approach consists in the method of handling nominals for atomic properties and decomposition in terms of components *encoded* in the translation. The reduction into $\mathrm{FO}^2[=]$ can be generalized to any relative logics provided,

1. the conditions[6] on the relations of the models can be expressed by a first-order formula involving at most two variables (see the definition of the formula $\mathtt{G}_0$ in Section 3.3), and
2. the class of binary relations underlying the logic is closed under intersection.

For instance, if in the definition of $\mathcal{L}$ we replace reflexivity by *weak reflexivity*, then decidability and finite model property still hold true[7]. This is particularly interesting since weakly reflexive and symmetric modal frames represent exactly the *negative* similarity relations in information systems (see e.g. [Vak91a,DO96]). For the sake of comparison, the class of reflexive and symmetric modal frames represent precisely the *positive* similarity relation in information systems.

We have also shown that $\mathcal{L}$-satisfiability is **EXPTIME**-hard (by taking advantage of the general results from [Hem96]), and that the problem can be solved by a deterministic Turing machine in time $O(2^{2^{2^{p(n)}}})$ for some polynomial $p(n)$, where n is the length of the tested formula. Indeed: the translation process $\mathtt{T}$ is exponential in time in the length of the formula, $\mathtt{T}$ may increase exponentially the length of the formula and satisfiability for $\mathrm{FO}^2[=]$ is in **NEXPTIME**. It is therefore an open problem to characterize more accurately the complexity class of $\mathcal{L}$-satisfiability. However, the translations we have established can already be used to mechanize the relative similarity logics by taking $\mathrm{FO}^2[=]$ as the target logic and by using a theorem prover dedicated to it. We conjecture that more efficient methods might exist for the mechanization.

References

[Bal97] Ph. Balbiani. Axiomatization of logics based on Kripke models with relative accessibility relations. In *[Oe97]*, pages 553–578, 1997.

[Ben83] J. van Benthem. *Modal logic and classical logic.* Bibliopolis, 1983.

[Ben98] J. van Benthem. The range of modal logic - An essay in memory of George Gargov. *Journal of Applied Non-Classical Logics*, 1998. To appear.

[Bla93] P. Blackburn. Nominal tense logic. *Notre Dame Journal of Formal Logic*, 34(1):56–83, 1993.

[Dem98] S. Demri. A logic with relative knowledge operators. *Journal of Logic, Language and Information*, 1998. To appear.

[DO96] S. Demri and E. Orłowska. Informational representability: Abstract models versus concrete models. In D. Dubois and H. Prade, editors, *Linz Seminar on Fuzzy Sets, Logics and Artificial Intelligence, Linz, Austria.* Kluwer Academic Publishers, February 1996. To appear.

[FdCO84] L. Fariñas del Cerro and E. Orłowska. DAL - A logic for data analysis. In T. O'Shea, editor, *ECAI-6*, pages 285–294, September 1984.

[GG93] G. Gargov and V. Goranko. Modal logic with names. *Journal of Philosophical Logic*, 22(6):607–636, December 1993.

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[6] If the conditions require more than two variables, the translation into $\mathrm{FO}^k[=]$ for some $k > 2$ is also possible. For instance, the logic defined in [Orło84a] can be translated into $\mathrm{FO}^3[=]$ by taking advantage of the results of the present paper.

[7] A binary relation R is *weakly reflexive* iff for any element x of some domain either $R(x) = \emptyset$ or $(x, x) \in R$.

[GKV97] E. Grädel, Ph. Kolaitis, and M. Vardi. On the decision problem for two-variable first-order logic. *Bulletin of Symbolic Logic*, 3(1):53–69, 1997.

[GP90] G. Gargov and S. Passy. A note on boolean modal logic. In P. Petkov, editor, *Summer School and Conference on Mathematical Logic '88*, pages 299–309. Plenum Press, 1990.

[Hem96] E. Hemaspaandra. The price of universality. *Notre Dame Journal of Formal Logic*, 37(2):173–203, 1996.

[Kon97] B. Konikowska. A logic for reasoning about relative similarity. *Studia Logica*, 58(1):185–226, 1997.

[Lew80] H. Lewis. Complexity results for classes of quanticational formulas. *Journal of Computer and System Sciences*, 21:317–353, 1980.

[Mor75] M. Mortimer. On language with two variables. *Zeit. für Math. Logik and Grund. der Math.*, 21:135–140, 1975.

[Oe97] E. Orlowska (ed.). *Incomplete Information: Rough Set Analysis.* Studies in Fuzziness and Soft Computing. Physica-Verlag, Heidelberg, 1997.

[OP84] E. Orłowska and Z. Pawlak. Representation of nondeterministic information. *Theoretical Computer Science*, 29:27–39, 1984.

[Orło84a] E. Orłowska. Logic of indiscernibility relations. In A. Skowron, editor, *5th Symposium on Computation Theory, Zaborów, Poland*, pages 177–186. LNCS 208, Springer-Verlag, 1984.

[Orło84b] E. Orłowska. Modal logics in the theory of information systems. *Zeitschr. f. Math. Logik und Grundlagen d. Math.*, 30(1):213–222, 1984.

[Paw81] Z. Pawlak. Information systems theoretical foundations. *Information Systems*, 6(3):205–218, 1981.

[PST97] L. Pacholski, W. Szwast, and L. Tendera. Complexity of two-variable logic with counting. In *LICS*, pages 318–327. IEEE, July 1997.

[PT91] S. Passy and T. Tinchev. An essay in combinatory dynamic logic. *Information and Computation*, 93:263–332, 1991.

[Vak91a] D. Vakarelov. Logical analysis of positive and negative similarity relations in property systems. In M. de Glas and D. Gabbay, editors, *First World Conference on the Fundamentals of Artificial Intelligence*, 1991.

[Vak91b] D. Vakarelov. A modal logic for similarity relations in Pawlak knowledge representation systems. *Fundamenta Informaticae*, 15:61–79, 1991.

[Vak91c] D. Vakarelov. Modal logics for knowledge representation systems. *Theoretical Computer Science*, 90:433–456, 1991.

[Var97] M. Vardi. Why is modal logic so robustly decidable? In *Descriptive complexity and finite models, A.M.S.*, 1997.