Natural Deduction for Hybrid Logic

TORBEN BRAÜNER, *Department of Computer Science, Roskilde University, P.O. Box 260, DK-4000 Roskilde, Denmark.*
E-mail: torben@ruc.dk

Abstract

In this paper we give a natural deduction formulation of hybrid logic. Our natural deduction system can be extended with additional inference rules corresponding to conditions on the accessibility relations expressed by so-called geometric theories. Thus, we give natural deduction systems in a uniform way for a wide class of hybrid logics which appears to be impossible in the context of ordinary modal logic. We prove soundness and completeness and we prove a normalization theorem. We finally prove a result which says that normal derivations in the natural deduction system correspond to derivations in a cut-free Gentzen system.

Keywords: Hybrid logic, modal logic, natural deduction, Gentzen systems.

1 Introduction

In this paper we give a natural deduction formulation of hybrid logic. Hybrid logic is obtained by adding to ordinary multi-modal logic further expressive power in the form of a second sort of propositional symbols called nominals, and moreover, by adding so-called satisfaction operators. A nominal is assumed to be true at exactly one world, so a nominal can be considered the name of a world. Thus, in hybrid logic a name is a particular sort of propositional symbol whereas in first-order logic it is an argument to a predicate. If a is a nominal and ϕ is an arbitrary formula, then a new formula $a : \phi$ called a satisfaction statement can be formed. The part $a :$ of $a : \phi$ is called a satisfaction operator. The satisfaction statement $a : \phi$ expresses that the formula ϕ is true at one particular world, namely the world at which the nominal a is true. Furthermore, we shall consider the so-called binders $\forall$ and $\downarrow$. The binders bind nominals to worlds in two different ways: the $\forall$ binder quantifies over worlds whereas $\downarrow$ binds a nominal to the actual world. The $\downarrow$ binder is definable in terms of $\forall$.

The history of hybrid logic goes back to Arthur Prior's work, more precisely, it goes back to what he called four grades of tense-logical involvement. They were presented in the book [22], Chapter XI (also Chapter XI in the new edition [17]). See also [21] Chapter V.6 and Appendix B.3–4. The stages progress from what can be regarded as pure first-order earlier–later logic to what can be regarded as pure tense logic; the goal being to be able to consider the tense logic of the fourth stage as encompassing the earlier–later logic of the first stage. In other words, the goal was to be able to translate the first-order logic of the earlier–later relation into tense logic. With this in mind, Prior introduced so-called instant-propositions:

> What I shall call the third grade of tense-logical involvement consists in treating the instant-variables a, b, c, etc. as also representing propositions ([22], p. 122–123).

In the context of modal logic, Prior called such propositions possible-world-propositions. Of course, this is what we in this paper call nominals. Prior also introduced the binder $\forall$ and what we here call satisfaction operators. See [18] for an account of Prior's work. The paper [8] makes a connection to Donald Davidson's notion of a theory of truth.

Natural deduction style inference rules for ordinary classical first-order logic were originally introduced by Gerhard Gentzen in [16] and later on considered by Dag Prawitz in [19, 20]. See [26] for a general introduction to natural deduction systems. In this paper we shall give natural deduction systems for various fragments of hybrid logic, namely the fragments of hybrid logic corresponding to each subset of the set of binders $\{\forall, \downarrow\}$ and we shall prove that the systems are sound and complete. Moreover, we shall show how to extend the natural deduction systems with additional inference rules corresponding to first-order conditions on the accessibility relations. The conditions we consider are expressed by so-called geometric theories. Different geometric theories give rise to different hybrid logics, so natural deduction systems for new hybrid logics can be obtained in a uniform way simply by adding inference rules as appropriate.

Natural deduction systems are characterized by having two different kinds of rules for each non-nullary connective; there is a kind of rule which introduces a connective (thus, the connective occurs in the conclusion, but not in the premisses) and there is a kind of rule which eliminates a connective (vice versa). A maximum formula in a derivation is then a formula occurrence which is both introduced by an introduction rule and eliminated by an elimination rule. Such a maximum formula can be considered a 'detour' in the derivation. Maximum formulas can be removed by using what is called proper reductions. Another kind of reduction, permutative reduction, is used to remove so-called permutable formulas. In our natural deduction systems, permutable formulas only occur in connection with the additional inference rules corresponding to geometric theories. A derivation is called normal if it contains no maximum or permutable formula and we prove a normalization theorem which says that any derivation can be rewritten to such a normal derivation by repeated applications of reductions. Normal derivations satisfiy a version of the subformula property called the quasi-subformula property.

The present paper is a revised and extended version of [7] which gave only a partial normalization theorem saying that any maximum or permutable formula can be removed provided it is not of the form $a : \Diamond c$. The problem not dealt with in [7] is that the removal of a maximum formula of that form in some cases generates new maximum formulas of the same form and the technique used in the standard normalization proof does not work in such cases since it requires that the generated maximum formulas all have a lower degree, that is, fewer connectives, than the original one. In the present paper this problem is solved by using the so-called $\Box$-graph of a derivation, which to the best of our knowledge is a novel notion.

Usually, when a modal natural deduction system is given, it is for one particuler modal logic. For example, Prawitz in [19] gives natural deduction systems for the modal logics $S4$ and $S5$. It has turned out to be difficult to formulate natural deduction systems for modal logics in a uniform way without introducing metalinguistic machinery. With reference to Prawitz's systems for $S4$ and $S5$, Robert Bull and Krister Segerberg note the following in their survey paper on modal logic in *Handbook of Philosophical Logic*.

> However, it has proved difficult to extend this sort of analysis to the great multitude of other systems of modal logic. It seems fair to say that a deductive treatment congenial to modal logic is yet to be found, for Hilbert systems are not suited for actual deduction, . . . ([13], p. 27–28)

So what is it that enables us to formulate uniform natural deduction rules for a wide class of hybrid logics when it appears to be impossible in the context of ordinary modal logic? Three features of hybrid logic appear to be crucial: we can express that

1. a formula ϕ is true at a world a, that is, $a : \phi$ is true;
2. a world a is R-related to a world c, that is, $a : \Diamond c$ is true; and
3. a world a is identical to a world c, that is, $a : c$ is true.

The first two features are exactly what is needed to formulate our introduction and elimination rules for a modal operator $\Box$ and the last two features are exactly what is needed to formulate our inference rules corresponding to geometric theories.

We finish the paper by setting our natural deduction system to work: we use our normalization theorem to prove result which says that normal derivations in the natural deduction system correspond to derivations in a cut-free Gentzen[1] system. This implies the completeness of the Gentzen system.

This paper is structured as follows. In the second section we recapitulate the basics of hybrid logic, in the third section we introduce our natural deduction systems, and in the fourth section we prove soundness and completeness. The fifth section is concerned with normalization and the sixth section is concerned with Gentzen systems corresponding to our natural deduction systems. In the final section we discuss some related work. A natural deduction system for first-order hybrid logic is given in the companion paper [10]. See also [9] where a functional completeness result is given for hybridized $S5$ which is a hybrid-logical version of $S5$.

2 Hybrid logic

In this section we recapitulate the basics of hybrid logic. We shall in many cases adopt the terminology of [4] and [1]. The hybrid logic we consider is obtained by adding a second sort of propositional symbols called *nominals* to ordinary multi-modal logic, that is, propositional logic extended with a finite number of modal operators $\Box_1, \ldots, \Box_m$. It is assumed that a set of ordinary propositional symbols and a countably infinite set of nominals are given. The sets are assumed to be disjoint. The metavariables $p, q, r, \ldots$ range over ordinary propositional symbols and $a, b, c, \ldots$ range over nominals.

Beside nominals, two so-called *binders*, $\forall$ and $\downarrow$, and for each nominal a, an operator $a :$ called a *satisfaction operator* are added. The formulas of hybrid modal logic are defined by the grammar

$$S ::= p \mid a \mid S \wedge S \mid S \rightarrow S \mid \bot \mid \Box_i S \mid a : S \mid \forall a S \mid \downarrow a S$$

where p is an ordinary propositional symbol and a is a nominal. In what follows, the metavariables $\phi, \psi, \theta, \ldots$ range over formulas. Negation, nullary conjunction, and disjunction are defined by the conventions that $\neg\phi$ is an abbreviation for $\phi \rightarrow \bot$, $\top$ is an abbreviation for $\neg\bot$, and $\phi \vee \psi$ is an abbreviation for $\neg(\neg\phi \wedge \neg\psi)$. Similarly, $\Diamond_i \phi$ is an abbreviation for $\neg\Box_i\neg\phi$ and $\exists a \phi$ is an abbreviation for $\neg\forall a \neg\phi$. It is assumed that we are working with a fixed number of modalities.

We now define frames and models. A *frame* is a tuple $(W, R_1, \ldots, R_m)$ where W is a non-empty set and $R_1, \ldots, R_m$ are binary relations on W. The elements of W are called *worlds* and each relation R_i is called an *accessibility relation*. A *model* is a tuple $(W, R_1, \ldots, R_m, V)$ where $(W, R_1, \ldots, R_m)$ is a frame and V is a *valuation*, that is, V is a function that to each pair consisting of an element of W and an ordinary propositional symbol assigns an element of $\{0, 1\}$. The model $(W, R_1, \ldots, R_m, V)$ is *based* on the frame $(W, R_1, \ldots, R_m)$. An *assignment* for a model $(W, R_1, \ldots, R_m, V)$ is a function g that to each nominal assigns an element of W. Given assignments g' and g, $g' \overset{a}{\sim} g$ means that g' agrees with g on all nominals save possibly a. Given a

[1] Note that there is danger of confusion here as Gentzen discovered natural deduction style as well as what is usually called Gentzen style, see [16].

model $\mathcal{M} = (W, R_1, \ldots, R_m, V)$, the relation $\mathcal{M}, g, w \models \phi$ is defined by induction, where g is an assignment, w is an element of W, and ϕ is a formula.

$$\begin{array}{rll}
\mathcal{M}, g, w \models p & \text{iff} & V(w,p) = 1 \\
\mathcal{M}, g, w \models a & \text{iff} & w = g(a) \\
\mathcal{M}, g, w \models \phi \wedge \psi & \text{iff} & \mathcal{M}, g, w \models \phi \text{ and } \mathcal{M}, g, w \models \psi \\
\mathcal{M}, g, w \models \phi \rightarrow \psi & \text{iff} & \mathcal{M}, g, w \models \phi \text{ implies } \mathcal{M}, g, w \models \psi \\
\mathcal{M}, g, w \models \bot & \text{iff} & \text{falsum} \\
\mathcal{M}, g, w \models \Box_i \phi & \text{iff} & \text{for any } v \in W \text{ such that } wR_iv, \mathcal{M}, g, v \models \phi \\
\mathcal{M}, g, w \models a : \phi & \text{iff} & \mathcal{M}, g, g(a) \models \phi \\
\mathcal{M}, g, w \models \forall a \phi & \text{iff} & \text{for any } g' \overset{a}{\sim} g, \mathcal{M}, g', w \models \phi \\
\mathcal{M}, g, w \models \downarrow a \phi & \text{iff} & \mathcal{M}, g', w \models \phi \text{ where } g' \overset{a}{\sim} g \text{ and } g'(a) = w.
\end{array}$$

A formula ϕ is said to be *true* at w if $\mathcal{M}, g, w \models \phi$; otherwise it is said to be *false* at w. By convention $\mathcal{M}, g \models \phi$ means $\mathcal{M}, g, w \models \phi$ for every element w of W and $\mathcal{M} \models \phi$ means $\mathcal{M}, g \models \phi$ for every assignment g. A formula ϕ is *valid* in a frame if and only if $\mathcal{M} \models \phi$ for any model $\mathcal{M}$ that is based on the frame in question. A formula ϕ is *valid* in a class of frames **F** if and only if ϕ is valid in any frame in **F**.

Now, let $\mathcal{O} \subseteq \{\downarrow, \forall\}$. In what follows $\mathcal{H}(\mathcal{O})$ denotes the fragment of hybrid logic in which the only binders are the binders in the set $\mathcal{O}$. It is assumed that the set $\mathcal{O}$ is fixed unless other is specified. If $\mathcal{O} = \{\downarrow\}$, then we simply write $\mathcal{H}(\downarrow)$, etc. The notions of free and bound occurrences of nominals are defined as in first-order logic. Also, if $\overline{a}$ is a list of pairwise distinct nominals and $\overline{c}$ is a list of nominals of the same length as $\overline{a}$, then $\psi[\overline{c}/\overline{a}]$ is the formula ψ where the nominals $\overline{c}$ have been simultaneously substituted for all free occurrences of the nominals $\overline{a}$. It is assumed that no nominal a_i in $\overline{a}$ occur free in ψ within the scope of $\forall c_i$ or $\downarrow c_i$.

2.1 Hybrid logic and first-order logic

It is a well-known observation that a model for hybrid logic can be considered a model for first-order logic and vice versa. The first-order language under consideration has a 1-place predicate symbol for each ordinary propositional symbol of modal logic, a 2-place predicate symbol for each modality, and a 2-place predicate symbol for equality. The language does not have constant or function symbols. The metavariables a, b, c, ... range over first-order variables. So the formulas of the first-order language we consider are defined by the grammar

$$S \ ::= \ p(a) \mid R_i(a,b) \mid a = b \mid S \wedge S \mid S \rightarrow S \mid \bot \mid \forall a S$$

where p is an ordinary propositional symbol of hybrid logic, and a and b are first-order variables. Note that the set of metavariables ranging over first-order variables is identical to the set of metavariables ranging over nominals. The connectives $\neg$, $\top$, $\vee$, and $\exists$ are defined in one of the usual ways. Free and bound variables are defined as usual. Also, if nominals of hybrid logic are identified with first-order variables, then an assignment in the sense of hybrid logic can be considered an assignment in the sense of first-order logic and vice versa. Given a model $\mathcal{M}$, considered a model for first-order logic, the relation $\mathcal{M} \models \phi[g]$ is defined by induction, cf. standard first-order logic, where g is an assignment and ϕ is a first-order formula. The formula ϕ is said to be *true* if $\mathcal{M} \models \phi[g]$; otherwise it is said to be *false*. By convention $\mathcal{M} \models \phi$ means $\mathcal{M} \models \phi[g]$ for every assignment g.

The observation that models and assignments for hybrid logic correspond to models and assignments for first-order logic gives rise to another observation, namely that the hybrid logic $\mathcal{H}(\forall)$ and first-order logic have the same expressive power in the sense that there exist truth-preserving translations in both directions. The following is a truth-preserving translation from first-order logic to the hybrid logic $\mathcal{H}(\forall)$.

$$\begin{array}{rcl} HT(p(a)) & = & a : p \\ HT(R_i(a,c)) & = & a : \Diamond_i c \\ HT(a = c) & = & a : c \\ HT(\phi \wedge \psi) & = & HT(\phi) \wedge HT(\psi) \\ HT(\phi \rightarrow \psi) & = & HT(\phi) \rightarrow HT(\psi) \\ HT(\bot) & = & \bot \\ HT(\forall a \phi) & = & \forall a HT(\phi). \end{array}$$

The history of the above mentioned observations goes back to the work of Arthur Prior, see for example [22], Chapter XI. Similarly, the hybrid logic $\mathcal{H}(\downarrow)$ has the same expressive power as the so-called *bounded fragment* of first-order logic. This was pointed out in [1].

3 Natural deduction for hybrid logic

In this section a natural deduction system for the hybrid logic $\mathcal{H}(\mathcal{O})$ is given, and moreover, we show how to extend the system with additional rules corresponding to conditions on the accessibility relations. Our natural deduction system shares several features with the tableau and Gentzen systems given in [3], see the final section of the present paper for a detailed comparison. Before giving the natural deduction system, we shall sketch the basics of natural deduction. See [19] and [26] for further details.

A characteristic feature of natural deduction is that derivations have the form of trees. Formula occurrences at the leaves of a tree are called *assumptions* and the formula occurrence at the root of the tree is called the *end-formula*. All assumptions are annotated with numbers. An assumption is either *undischarged* or *discharged*. If an assumption is discharged, then it is discharged at a specified rule-instance which is indicated by annotating the assumption and the rule-instance with identical numbers. We shall often omit this information when no confusion can occur. A rule-instance annotated with some number discharges all undischarged assumptions that are above it and are annotated with the number in question, and moreover, are occurrences of a specified formula. Two assumptions in a derivation belong to the same *parcel* if they are annotated with the same number and are occurrences of the same formula, and moreover, either are both undischarged or have both been discharged at the same rule-instance. Thus, in this terminology rules discharge parcels. We shall make use of the standard notation

$$\begin{array}{c} [\phi^r] \\ \vdots\ \pi \\ \psi \end{array} \qquad\qquad \begin{array}{c} (\phi^r) \\ \vdots\ \pi \\ \psi \end{array} \qquad\qquad \begin{array}{c} \vdots\ \tau \\ \phi \\ \vdots\ \pi \\ \psi \end{array}$$

which from left to right means: (i) a derivation π where ψ is the end-formula and $[\phi^r]$ is the parcel consisting of all undischarged assumptions that have the form ϕ^r; (ii) a derivation π where ψ is the end-formula and (ϕ^r) is a single undischarged assumption of the form ϕ^r; and (iii) a derivation π where ψ is the end-formula and a derivation τ with end-formula ϕ has been substituted for all the undischarged assumptions indicated by either $[\phi^r]$ or (ϕ^r). A derivation in a natural deduction system is generated from a set of inference rules from derivations consisting of a single undischarged assumption.

$$\frac{}{a:a}\,(Ref) \qquad\qquad \frac{a:c \qquad a:\phi}{c:\phi}\,(Nom1)^{*} \qquad\qquad \frac{a:c \qquad a:\Diamond_i b}{c:\Diamond_i b}\,(Nom2)$$

$*$ ϕ is a propositional symbol (ordinary or a nominal).

FIGURE 2. Natural deduction rules for nominals

Another characteristic feature of natural deduction is that there are two different kinds of rules for each non-nullary connective; there is a kind of rule called *introduction rule* which introduces a connective (that is, the connective occurs in the conclusion of the rule, but not in the premisses) and there is a kind of rule called *elimination rule* which eliminates a connective (the connective occurs in the premiss of the rule, but not in the conclusion). Introduction rules traditionally have names of the form $(\ldots I \ldots)$, and similarly, elimination rules traditionally have names of the form $(\ldots E \ldots)$.

We shall make use of the following conventions. The metavariables π, τ, ... range over derivations. Formulas of the form $a : \phi$ are called *satisfaction statements*, cf. a similar notion in [3]. The metavariables Γ, Δ, ... range over sets of satisfaction statements. A derivation π is a *derivation of* ϕ if the end-formula of π is an occurrence of ϕ and π is a *derivation from* Γ if each undischarged assumption in π is an occurrence of a formula in Γ (note that numbers annotating undischarged assumptions are ignored). If there exists a derivation of ϕ from $\emptyset$, then we simply say that ϕ is *derivable*. Moreover, $\pi[\overline{c}/\overline{a}]$ is the derivation π where each formula occurrence ψ has been replaced by $\psi[\overline{c}/\overline{a}]$.

Now, natural deduction inference rules for hybrid logic are given in Figure 1 and Figure 2. All formulas in the rules are satisfaction statements. Note that the rules $(\bot 1)$ and $(\bot 2)$ are neither introduction rules nor elimination rules (recall that $\neg\phi$ is an abbreviation for $\phi \rightarrow \bot$). Our natural deduction system for $\mathcal{H}(\mathcal{O})$ is obtained from the rules given in Figure 1 and Figure 2 by leaving out the rules for the binders that are not in the set $\mathcal{O}$ (recall that $\mathcal{O} \subseteq \{\downarrow, \forall\}$). The system thus obtained will be denoted $\mathbf{N}_{\mathcal{H}(\mathcal{O})}$. So for example $\mathbf{N}_{\mathcal{H}(\forall)}$ is obtained by leaving out the rules $(\downarrow I)$ and $(\downarrow E)$. Below is an example of a derivation in $\mathbf{N}_{\mathcal{H}()}$.

$$\cfrac{\cfrac{\cfrac{\cfrac{a:\Box_i\neg\phi^{1} \qquad \cfrac{a:(\Diamond_i c \wedge c:\phi)^{2}}{a:\Diamond_i c}\,(\wedge E1)}{c:\neg\phi}\,(\Box_i E) \qquad \cfrac{\cfrac{a:(\Diamond_i c \wedge c:\phi)^{2}}{a:c:\phi}\,(\wedge E2)}{c:\phi}\,(:E)}{c:\bot}\,(\rightarrow E)}{a:\bot}\,(\bot 2)}{\cfrac{a:\Diamond_i\phi}{a:((\Diamond_i c \wedge c:\phi)\rightarrow \Diamond_i\phi)}\,(\rightarrow I)^{2}}\,(\rightarrow I)^{1}$$

The end-formula of the derivation is the axiom *Bridge* given in [1] prefixed by a satisfaction operator.

The natural deduction system $\mathbf{N}_{\mathcal{H}(\mathcal{O})}$ corresponds to the class of all frames, that is, the class of frames where no conditions are imposed on the accessibility conditions. Hence, it is a hybrid version of the standard modal logic K.

$$\dfrac{s_1 \quad \dots \quad s_n \qquad \begin{array}{c}[s_{11}]\dots[s_{1n_1}]\\ \vdots\\ \phi\end{array} \qquad \dots \qquad \begin{array}{c}[s_{m1}]\dots[s_{mn_m}]\\ \vdots\\ \phi\end{array}}{\phi}\,(R_\theta)^*$$

$*$ None of the nominals in $\overline{c}$ occur free in ϕ or in any of the undischarged assumptions other than the specified occurrences of s_{jk}. (Recall that nominals are identified with first-order variables and that $\overline{c}$ are the first-order variables existentially quantified over in the formula θ.)

FIGURE 3. Natural deduction rules for geometric theories

1. Symmetry	$\forall a \forall c(R_i(a,c) \rightarrow R_i(c,a))$
2. Antisymmetry	$\forall a \forall c((R_i(a,c) \wedge R_i(c,a)) \rightarrow a = c)$
3. Irreflexivity	$\forall a(R_i(a,a) \rightarrow \bot)$
4. Directedness	$\forall a \forall b \forall c((R_i(a,b) \wedge R_i(a,c)) \rightarrow \exists d(R_i(b,d) \wedge R_i(c,d)))$

FIGURE 4. A sample of conditions on R_i

$$\dfrac{a : \Diamond_i c \qquad \begin{array}{c}[c : \Diamond_i a]\\ \vdots\\ \phi\end{array}}{\phi}\,(R_{\theta_1}) \qquad\qquad \dfrac{a : \Diamond_i c \quad c : \Diamond_i a \qquad \begin{array}{c}[a : c]\\ \vdots\\ \phi\end{array}}{\phi}\,(R_{\theta_2})$$

$$\dfrac{a : \Diamond_i a}{\phi}\,(R_{\theta_3}) \qquad\qquad \dfrac{a : \Diamond_i b \quad a : \Diamond_i c \qquad \begin{array}{c}[b : \Diamond_i d][c : \Diamond_i d]\\ \vdots\\ \phi\end{array}}{\phi}\,(R_{\theta_4})^*$$

$*$ d does not occur free in ϕ or in any undischarged assumptions other than the specified occurrences of $c : \Diamond_i d$ and $b : \Diamond_i d$.

FIGURE 5. Rules corresponding to conditions on R_i

3.1 Conditions on the accessibility relations

In what follows we shall consider natural deduction systems obtained by extending $\mathbf{N}_{\mathcal{H}(\mathcal{O})}$ with additional inference rules corresponding to first-order conditions on the accessibility relations. The conditions we consider are expressed by so-called geometric theories. A first-order formula is *geometric* if it is built out of atomic formulas of the forms $R_i(a, c)$ and $a = c$ using only the connectives $\bot$, $\wedge$, $\vee$, and $\exists$. In what follows, the metavariables S_k and S_{jk} range over atomic first-order formulas of the forms $R_i(a, c)$ and $a = c$. See [28] for an introduction to geometric logic.

Now, a *geometric theory* is a finite set of closed first-order formulas each having the form $\forall\overline{a}(\phi \rightarrow \psi)$ where the formulas ϕ and ψ are geometric, $\overline{a}$ is a list $a_1, \ldots, a_l$ of variables, and $\forall\overline{a}$ is an abbreviation for $\forall a_1 \ldots \forall a_l$. It can be proved, cf. [25], that any geometric theory is equivalent to a *basic geometric theory*, which is a geometric theory in which each formula has the form

$$\forall\overline{a}((S_1 \wedge \ldots \wedge S_n) \rightarrow \exists\overline{c} \bigvee_{j=1}^{m} (S_{j1} \wedge \ldots \wedge S_{jn_j}))$$

where $n, m \geq 0$ and $n_1, \ldots, n_m \geq 1$. For simplicity, we assume that the variables in the list $\overline{a}$ are pairwise distinct, that the variables in $\overline{c}$ are pairwise distinct, and that no variable occurs in both $\overline{c}$ and $\overline{a}$. A sample of formulas of the form displayed above is given in Figure 4. Note that such a formula is a Horn clause if $\overline{c}$ is empty, $m = 1$, and $n_m = 1$. Thus, the first two formulas in Figure 4 are Horn clauses. Also, note that the third formula in Figure 4 is identical to $\forall a \neg R_i(a, a)$.

In what follows, the metavariables s_k and s_{jk} range over hybrid-logical formulas of the forms $a : \Diamond_i c$ and $a : c$. It turns out that basic geometric theories correspond to straightforward natural deduction rules for hybrid logic: with a formula θ of the form displayed above, we associate the natural deduction rule (R_θ) given in Figure 3 where s_k is of the form $HT(S_k)$ and s_{jk} is of the form $HT(S_{jk})$ (HT is the translation from first-order logic to hybrid logic given in the previous section). If θ_1, …, θ_4 are formulas of the forms given in Figure 4, then the associated natural deduction rules (R_{θ_1}), …, (R_{θ_4}) are the rules in Figure 5. Note that the rule (R_{θ_3}) has zero non-relational premisses. Now, let $\mathbf{T}$ be any basic geometric theory. The natural deduction system obtained by extending $\mathbf{N}_{\mathcal{H}(\mathcal{O})}$ with the set of rules $\{(R_\theta) \mid \theta \in \mathbf{T}\}$ will be denoted $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$. We shall assume that we are working with a fixed basic geometric theory $\mathbf{T}$ unless other is specified.

It is straightforward to check that if a formula θ of the form displayed above is a Horn clause, then the rule (R_θ) given in Figure 3 can be replaced by the simpler rule below (which we also have called (R_θ)).

$$\frac{s_1 \quad \ldots \quad s_n}{s_{11}} \; (R_\theta)$$

Natural deduction rules corresponding to Horn clauses were discussed in [20].

3.2 Some eliminable rules

Below we shall prove a small proposition regarding some eliminable rules. To this end we need a convention: The *degree* of a formula is the number of occurrences of non-nullary connectives in it.

PROPOSITION 3.1
The rules

$$\dfrac{\begin{matrix}[a:\neg\phi]\\ \vdots \\ a:\bot\end{matrix}}{a:\phi}\,(\bot) \qquad\qquad \dfrac{a:c \quad a:\phi}{c:\phi}\,(Nom)$$

are eliminable in $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$.

PROOF. The proof that $(\bot)$ is eliminable is along the lines of a similar proof for ordinary classical first-order logic given in [19]. The proof that (Nom) is eliminable is analogous. ■

Note in the proposition above that ϕ can be any formula; not just a propositional symbol. Thus, the rule $(\bot)$ generalizes the rule $(\bot 1)$ whereas (Nom) generalizes $(Nom1)$ (and the rule $(Nom2)$ as well). The side-conditions on the rules $(\bot 1)$ and $(Nom1)$ enable us to prove a normalization theorem such that normal derivations satisfy a version of the subformula property called the quasi-subformula property. We shall return to this issue later.

4 Soundness and completeness

The aim of this section is to prove soundness and completeness. We shall need the standard substitution lemma below.

LEMMA 4.1 (Substitution lemma)
Let $\mathcal{M}$ be a model and let ψ be a formula. For any world w and any assignments g and g' such that $g(a) = g'(c)$ and $g \overset{a}{\sim} g'$, $\mathcal{M}, g, w \models \psi$ if and only if $\mathcal{M}, g', w \models \psi[c/a]$.

PROOF. Straightforward induction on the structure of ψ. ■

Recall that we are working with a fixed basic geometric theory $\mathbf{T}$. A model $\mathcal{M}$ is called a $\mathbf{T}$-*model* if and only if $\mathcal{M} \models \theta$ for every formula $\theta \in \mathbf{T}$ (note that the model $\mathcal{M}$ in this definition is considered a first-order model).

THEOREM 4.2 (Soundness)
The first statement below implies the second statement.

1. ψ is derivable from Γ in $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$.
2. For any $\mathbf{T}$-model $\mathcal{M}$ and any assignment g, if, for any formula $\theta \in \Gamma$, $\mathcal{M}, g \models \theta$, then $\mathcal{M}, g \models \psi$.

PROOF. Induction on the structure of the derivation of ψ. ■

In what follows, we shall prove completeness. The proof we give is similar to the completeness proof in [3]. However, we use maximal consistent sets instead of Hintikka sets. Also, our proof is in some ways similar to the completeness proof in [2].

DEFINITION 4.3
A set of satisfaction statements Γ in $\mathcal{H}(\mathcal{O})$ is $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$-*inconsistent* if and only if $a : \bot$ is derivable from Γ in $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ for some nominal a and Γ is $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$-*consistent* if and only if Γ is not $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$-inconsistent. Moreover, Γ is *maximal* $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$-*consistent* if and only if Γ is $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$-consistent and any set of satisfaction statements in $\mathcal{H}(\mathcal{O})$ that properly extends Γ is $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$-inconsistent.

We shall frequently omit the reference to $\mathcal{H}(\mathcal{O})$ and $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ where no confusion can occur. The definition above leads us to the lemma below.

LEMMA 4.4
If a set of satisfaction statements Γ is consistent, then for every satisfaction statement $a : \phi$, either $\Gamma \cup \{a : \phi\}$ is consistent or $\Gamma \cup \{a : \neg\phi\}$ is consistent.

PROOF. Straightforward. ■

The Lindenbaum lemma below is similar to the Lindenbaum lemma in [2].

LEMMA 4.5 (Lindenbaum lemma)
Let $\overline{\mathcal{H}(\mathcal{O})}$ be the hybrid logic obtained by extending the set of nominals in $\mathcal{H}(\mathcal{O})$ with a countably infinite set of new nominals. Let $\phi_1, \phi_2, \phi_3, \ldots$ be an enumeration of all satisfaction statements in $\overline{\mathcal{H}(\mathcal{O})}$. For every $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$-consistent set of satisfaction statements Γ, a maximal $\mathbf{N}_{\overline{\mathcal{H}(\mathcal{O})}} + \mathbf{T}$-consistent set of satisfaction statements $\Gamma^* \supseteq \Gamma$ is defined as follows. First, Γ^0 is defined to be Γ. Second, Γ^{n+1} is defined by induction. If $\Gamma^n \cup \{\phi_{n+1}\}$ is $\mathbf{N}_{\overline{\mathcal{H}(\mathcal{O})}} + \mathbf{T}$-inconsistent, then Γ^{n+1} is defined to be Γ^n. Otherwise Γ^{n+1} is defined to be

1. $\Gamma^n \cup \{\phi_{n+1}, b : \psi, a : \Diamond_i b\}$ if ϕ_{n+1} is of the form $a : \Diamond_i \psi$;
2. $\Gamma^n \cup \{\phi_{n+1}, b : \psi[b/c], a : b\}$ if ϕ_{n+1} is of the form $a : \downarrow c\psi$;
3. $\Gamma^n \cup \{\phi_{n+1}, a : \psi[b/c]\}$ if ϕ_{n+1} is of the form $a : \exists c\psi$;
4. $\Gamma^n \cup \{\phi_{n+1}, e : \vee_{j=1}^m (s_{j1} \wedge \ldots \wedge s_{jn_j})[\overline{d}, \overline{b}/\overline{a}, \overline{c}]\}$ if there exists a formula in $\mathbf{T}$ of the form $\forall\overline{a}((S_1 \wedge \ldots \wedge S_n) \to \exists\overline{c} \vee_{j=1}^m (S_{j1} \wedge \ldots \wedge S_{jn_j}))$ such that $m \geq 1$ and $\phi_{n+1} = e : (s_1 \wedge \ldots \wedge s_n)[\overline{d}/\overline{a}]$ for some nominals $\overline{d}$ and e; and
5. $\Gamma^n \cup \{\phi_{n+1}\}$ if none of the clauses above apply.

In clause 1, 2, and 3, b is a new nominal that does not occur in Γ^n or ϕ_{n+1}, and similarly, in clause 4, $\overline{b}$ is a list of new nominals of the same length as $\overline{c}$ such that none of the nominals in b occur in Γ^n or ϕ_{n+1}. Finally, Γ^* is defined to be $\cup_{n \geq 0} \Gamma^n$.

PROOF. First, Γ^0 is $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$-consistent by definition and hence also $\mathbf{N}_{\overline{\mathcal{H}(\mathcal{O})}} + \mathbf{T}$-consistent. Secondly, to check that the consistency of Γ^n implies the consistency of Γ^{n+1}, we need to check the first four clauses in the definition of Γ^{n+1}.

- If ϕ_{n+1} is of the form $a : \Diamond_i \psi$, then assume conversely that $f : \bot$ is derivable from $\Gamma^n \cup \{\phi_{n+1}, b : \psi, a : \Diamond_i b\}$. Then $b : \neg\psi$ is derivable from $\Gamma^n \cup \{\phi_{n+1}, a : \Diamond_i b\}$ wherefore $a : \Box_i \neg\psi$ is derivable from $\Gamma^n \cup \{\phi_{n+1}\}$ by the rule $(\Box_i I)$. But then $a : \bot$ is derivable from $\Gamma^n \cup \{\phi_{n+1}\}$ as $\phi_{n+1} = a : \neg\Box_i \neg\psi$.
- If ϕ_{n+1} is of the form $a : \downarrow c\psi$, then assume conversely that $f : \bot$ is derivable from $\Gamma^n \cup \{\phi_{n+1}, b : \psi[b/c], a : b\}$. Then $b : \neg\psi[b/c]$ is derivable from $\Gamma^n \cup \{\phi_{n+1}, a : b\}$ wherefore $a : \downarrow c\neg\psi$ is derivable from $\Gamma^n \cup \{\phi_{n+1}\}$ by the rule $(\downarrow I)$. But $a : (\downarrow c\neg\psi \to \neg \downarrow c\psi)$ is derivable, thus $a : \neg \downarrow c\psi$ and hence also $a : \bot$ is derivable from $\Gamma^n \cup \{\phi_{n+1}\}$.
- The case involving $\exists$ is similar to the case involving $\Diamond_i$.
- If there exists a formula in $\mathbf{T}$ of the form $\forall\overline{a}((S_1 \wedge \ldots \wedge S_n) \to \exists\overline{c} \vee_{j=1}^m (S_{j1} \wedge \ldots \wedge S_{jn_j}))$ such that $m \geq 1$ and $\phi_{n+1} = e : (s_1 \wedge \ldots \wedge s_n)[\overline{d}/\overline{a}]$ for some nominals $\overline{d}$ and e, then assume conversely that $f : \bot$ is derivable from $\Gamma^n \cup \{\phi_{n+1}, e : \vee_{j=1}^m (s_{j1} \wedge \ldots \wedge s_{jn_j})[\overline{d}, \overline{b}/\overline{a}, \overline{c}]\}$. Then $e : \wedge_{j=1}^m \neg(s_{j1} \wedge \ldots \wedge s_{jn_j})[\overline{d}, \overline{b}/\overline{a}, \overline{c}]$ is derivable from $\Gamma^n \cup \{\phi_{n+1}\}$, and hence, $e : \bot$ is derivable from $\Gamma^n \cup \{\phi_{n+1}, s_{j1}[\overline{d}, \overline{b}/\overline{a}, \overline{c}], \ldots, s_{jn_j}[\overline{d}, \overline{b}/\overline{a}, \overline{c}]\}$ for any j where $1 \leq j \leq m$.

But $s_1[\overline{d}/\overline{a}]$, ..., $s_n[\overline{d}/\overline{a}]$ are derivable from $\Gamma^n \cup \{\phi_{n+1}\}$. Therefore $e : \bot$ is derivable from $\Gamma^n \cup \{\phi_{n+1}\}$ by the rule (R_θ).

We conclude that each Γ^n is consistent, which trivially implies that Γ^* is consistent. We now just need to prove that Γ^* is maximal consistent. Assume conversely that there exists a satisfaction statement $a : \phi$ such that $a : \phi \notin \Gamma^*$ as well as $a : \neg\phi \notin \Gamma^*$, cf. Lemma 4.4. Then $\phi_p \notin \Gamma^p$ and $\phi_q \notin \Gamma^q$ where $\phi_p = a : \phi$ and $\phi_q = a : \neg\phi$. So $\Gamma^{p-1} \cup \{\phi_p\}$ is inconsistent and so is $\Gamma^{q-1} \cup \{\phi_q\}$. If $p < q$, then $\Gamma^{p-1} \subseteq \Gamma^{q-1}$ wherefore $\Gamma^{q-1} \cup \{\phi_p\}$ is inconsistent. Thus, Γ^{q-1} is inconsistent by Lemma 4.4. The argument is analogous if $q < p$. ■

If $\forall \notin \mathcal{O}$, then ϕ_{n+1} in the lemma above can obviously not be of the form $a : \exists c\psi$, so the parts of the definition of Γ^{n+1} involving that case are superfluous. An analogous remark applies if $\downarrow \notin \mathcal{O}$. Below we shall define a canonical model. First a small lemma.

LEMMA 4.6
Let Δ be a maximal $\mathbf{N}_{\overline{\mathcal{H}(\mathcal{O})}} + \mathbf{T}$-consistent set of satisfaction statements. Let $\sim_\Delta$ be the binary relation on the set of nominals of $\overline{\mathcal{H}(\mathcal{O})}$ defined by the convention that $a \sim_\Delta a'$ if and only if $a : a' \in \Delta$. Then the relation $\sim_\Delta$ is an equivalence relation with the following properties.

1. If $a \sim_\Delta a'$, $c \sim_\Delta c'$, and $a : \Diamond_i c \in \Delta$, then $a' : \Diamond_i c' \in \Delta$.
2. If $a \sim_\Delta a'$ and $a : p \in \Delta$, then $a' : p \in \Delta$.

PROOF. It follows straightforwardly from Lemma 4.4 and the rules (Ref) and $(\mathit{Nom1})$ that $\sim_\Delta$ is reflexive, symmetric, and transitive. The first mentioned property follows from Lemma 4.4, the rule $(\mathit{Nom2})$, and the observation that $a' : \Diamond_i c'$ is derivable from $\{a' : \Diamond_i c, c : c'\}$. The second property follows from Lemma 4.4 and the rule $(\mathit{Nom1})$. ■

Given a nominal a, we let $[a]$ denote the equivalence class of a with respect to $\sim_\Delta$. We now define a canonical model.

DEFINITION 4.7 (Canonical model)
Let Δ be a maximal $\mathbf{N}_{\overline{\mathcal{H}(\mathcal{O})}} + \mathbf{T}$-consistent set of satisfaction statements. A model $\mathcal{M}^\Delta = (W^\Delta, R_1^\Delta, \ldots, R_m^\Delta, V^\Delta)$ and an assignment g^Δ for $\mathcal{M}^\Delta$ is defined as follows.

- $W^\Delta = \{[a] \mid a \text{ is a nominal of } \overline{\mathcal{H}(\mathcal{O})}\}$.
- $[a]R_i^\Delta[c]$ if and only if $a : \Diamond_i c \in \Delta$.
- $V^\Delta([a], p) = \begin{cases} 1 & \text{if } a : p \in \Delta. \\ 0 & \text{otherwise.} \end{cases}$
- $g^\Delta(a) = [a]$.

Note that the first property of $\sim_\Delta$ mentioned in Lemma 4.6 implies that R_i^Δ is well-defined, and similarly, the second property implies that V^Δ is well-defined. Given the Lindenbaum lemma and the definition of a canonical model, we just need one small lemma before we are ready to prove a truth lemma.

LEMMA 4.8
Let ϕ be a satisfaction statement of the hybrid logic $\mathcal{H}(\mathcal{O})$, and let c and b be nominals such that b does not occur in ϕ. Let ϕ' be ϕ where each occurrence of c that is not free has been replaced by b. Then ϕ' is derivable from $\{\phi\}$ and ϕ is derivable from $\{\phi'\}$ in $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \emptyset$.

PROOF. Induction on the degree of ϕ. ■

For example, the satisfaction statement $d : \forall b(b : p)$ of $\mathcal{H}(\forall)$ is derivable from $\{d : \forall c(c : p)\}$. Now the truth lemma.

LEMMA 4.9 (Truth lemma)
Let Γ be a $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$-consistent set of satisfaction statements. Then for any satisfaction statement $a : \phi$, $a : \phi \in \Gamma^*$ if and only if $\mathcal{M}^{\Gamma^*}, g^{\Gamma^*}, [a] \models \phi$.

PROOF. Induction on the degree of ϕ. We consider only two cases; the other cases are similar.

The first case is where ϕ is of the form $\Box_i\theta$. Assume that $a : \Box_i\theta \in \Gamma^*$. We then have to prove that $\mathcal{M}^{\Gamma^*}, g^{\Gamma^*}, [c] \models \theta$ for any nominal c such that $[a]R_i^{\Gamma^*}[c]$, that is, such that $a : \Diamond_i c \in \Gamma^*$. But $a : \Diamond_i c \in \Gamma^*$ implies $c : \theta \in \Gamma^*$ by the rule $(\Box_i E)$ and this implies $\mathcal{M}^{\Gamma^*}, g^{\Gamma^*}, [c] \models \theta$ by induction. On the other hand, assume that $\mathcal{M}^{\Gamma^*}, g^{\Gamma^*}, [a] \models \Box_i\theta$, that is, $\mathcal{M}^{\Gamma^*}, g^{\Gamma^*}, [c] \models \theta$ for any nominal c such that $a : \Diamond_i c \in \Gamma^*$. Now, if $a : \neg\Box_i\theta \in \Gamma^*$, then also $a : \Diamond_i\neg\theta \in \Gamma^*$ as $a : (\neg\Box_i\theta \rightarrow \Diamond_i\neg\theta)$ is derivable. Therefore by definition of Γ^*, there exists a nominal b such that $b : \neg\theta \in \Gamma^*$ and $a : \Diamond_i b \in \Gamma^*$. But then $\mathcal{M}^{\Gamma^*}, g^{\Gamma^*}, [b] \models \theta$ by assumption and hence $b : \theta \in \Gamma^*$ by induction. Thus, we conclude that $a : \neg\Box_i\theta \notin \Gamma^*$ and hence $a : \Box_i\theta \in \Gamma^*$ by Lemma 4.4.

The second case we consider is where ϕ is of the form $\downarrow c\theta$. Assume that $a :\downarrow c\theta \in \Gamma^*$. We then have to prove that $\mathcal{M}^{\Gamma^*}, g, [a] \models \theta$ where $g \overset{c}{\sim} g^{\Gamma^*}$ and $g(c) = [a]$. Let θ' be θ where each occurrence of a that is not free has been replaced by some nominal that does not occur in $a : \theta$. Then $a :\downarrow c\theta' \in \Gamma^*$ as $a : (\downarrow c\theta \rightarrow \downarrow c\theta')$ is derivable by Lemma 4.8. So $a : \theta'[a/c] \in \Gamma^*$ by the rules $(\downarrow E)$ and (Ref). By induction we get $\mathcal{M}^{\Gamma^*}, g^{\Gamma^*}, [a] \models \theta'[a/c]$ and therefore $\mathcal{M}^{\Gamma^*}, g, [a] \models \theta'$ by Lemma 4.1. But $a : (\theta' \rightarrow \theta)$ is derivable by Lemma 4.8 and therefore valid by Theorem 4.2, so $\mathcal{M}^{\Gamma^*}, g, [a] \models \theta$. On the other hand, assume that $\mathcal{M}^{\Gamma^*}, g^{\Gamma^*}, [a] \models \downarrow c\theta$. If $a : \neg\downarrow c\theta \in \Gamma^*$, then also $a :\downarrow c\neg\theta \in \Gamma^*$ as $a : (\neg\downarrow c\theta \rightarrow \downarrow c\neg\theta)$ is derivable. Therefore by definition of Γ^*, there exists a nominal b such that $b : \neg\theta[b/c] \in \Gamma^*$ and $a : b \in \Gamma^*$. Now, let $g \overset{c}{\sim} g^{\Gamma^*}$ such that $g(c) = [a]$. Then by assumption $\mathcal{M}^{\Gamma^*}, g, [a] \models \theta$ and hence $\mathcal{M}^{\Gamma^*}, g^{\Gamma^*}, [a] \models \theta[b/c]$ by Lemma 4.1 as $[a] = [b]$ since $a \sim_{\Gamma^*} b$. Therefore $b : \theta[b/c] \in \Gamma^*$ by induction. We conclude that $a : \neg\downarrow c\theta \notin \Gamma^*$ and hence $a :\downarrow c\theta \in \Gamma^*$ by Lemma 4.4. ∎

The treatment of $\downarrow$ in the lemma above is similar to the treatment of the binder $\exists$ in the truth lemma of [6]. We now just need one lemma before we can prove completeness.

LEMMA 4.10
Let Γ be a $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$-consistent set of satisfaction statements. Then the model $\mathcal{M}^{\Gamma^*}$ is a $\mathbf{T}$-model.

PROOF. Let $\theta \in \mathbf{T}$. Then θ has the form $\forall\overline{a}((S_1 \wedge \ldots \wedge S_n) \rightarrow \exists\overline{c} \vee_{j=1}^{m} (S_{j1} \wedge \ldots \wedge S_{jn_j}))$ where $\overline{a} = a_1, \ldots, a_l$. Assume that g is an assignment for $\mathcal{M}^{\Gamma^*}$ such that $\mathcal{M}^{\Gamma^*} \models S_1[g]$, ..., $\mathcal{M}^{\Gamma^*} \models S_n[g]$. (Note that $\mathcal{M}^{\Gamma^*}$ here is considered a first-order model.) Let $g(a_1) = [d_1]$, ..., $g(a_l) = [d_l]$. Then $s_1[\overline{d}/\overline{a}], \ldots, s_n[\overline{d}/\overline{a}] \in \Gamma^*$ by Lemma 4.9. If $m \geq 1$, then by definition of Γ^* there exists a list of nominals $\overline{b}$ such that $e : \vee_{j=1}^{m}(s_{j1} \wedge \ldots \wedge s_{jn_j})[\overline{d}, \overline{b}/\overline{a}, \overline{c}] \in \Gamma^*$ since $e : (s_1 \wedge \ldots \wedge s_n)[\overline{d}/\overline{a}] \in \Gamma^*$ where e is an arbitrary nominal. Therefore $e : (s_{j1} \wedge \ldots \wedge s_{jn_j})[\overline{d}, \overline{b}/\overline{a}, \overline{c}] \in \Gamma^*$ and hence $s_{j1}[\overline{d}, \overline{b}/\overline{a}, \overline{c}], \ldots, s_{jn_j}[\overline{d}, \overline{b}/\overline{a}, \overline{c}] \in \Gamma^*$ for some j where $1 \leq j \leq m$. But then it follows from Lemma 4.9 that $\mathcal{M}^{\Gamma^*} \models \exists\overline{c} \vee_{j=1}^{m} (S_{j1} \wedge \ldots \wedge S_{jn_j})[g]$. On the other hand, if $m = 0$, then $e : \bot \in \Gamma^*$ by the rule (R_θ) which contradicts the consistency of Γ^*. ∎

Now the completeness theorem.

THEOREM 4.11 (Completeness)
The second statement below implies the first statement.

1. ψ is derivable from Γ in $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$.

2. For any $\mathbf{T}$-model $\mathcal{M}$ and any assignment g, if, for any formula $\theta \in \Gamma$, $\mathcal{M}, g \models \theta$, then $\mathcal{M}, g \models \psi$.

PROOF. We are done if Γ is inconsistent, cf. Proposition 3.1. So assume that Γ is consistent. Now, assume that ψ is not derivable from Γ and let $\psi = a : \phi$. Then $\Gamma \cup \{a : \neg\phi\}$ is consistent. Let $\Delta = (\Gamma \cup \{a : \neg\phi\})^*$, cf. Lemma 4.5, and consider the model $\mathcal{M}^\Delta$ and the assignment g^Δ. By Lemma 4.9, $\mathcal{M}^\Delta, g^\Delta \models \theta$ for any formula $\theta \in \Gamma$, and also $\mathcal{M}^\Delta, g^\Delta \models a : \neg\phi$. But this contradicts the second statement in the theorem since $\mathcal{M}^\Delta$ is a $\mathbf{T}$-model by Lemma 4.10. ■

5 Normalization

In this section we give reduction rules for the natural deduction system $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ and we prove a normalization theorem. First some conventions. If a premiss of a rule has the form $a : c$ or $a : \Diamond_i c$, then it is called a *relational premiss*, and similarly, if the conclusion of a rule has the form $a : c$ or $a : \Diamond_i c$, then it is called a *relational conclusion*. Moreover, if an assumption discharged by a rule has the form $a : c$ or $a : \Diamond_i c$, then it is called a *relationally discharged assumption*. The premiss of the form $a : \phi$ in the rule $(\to E)$ is called *minor*. A premiss of an elimination rule that is neither minor nor relational is called *major*. Note that the notion of a relational premiss is defined in terms of rules; not rule-instances. A similar remark applies to the other notions above. Thus, a formula occurrence in a derivation might be of the form $a : \Diamond_i c$ and also be the major premiss of an instance of $(\to E)$. Note that the premisses $s_1, \ldots, s_n$ in a (R_θ) rule are relational and that all the assumptions discharged by such a rule are relationally discharged.

A *maximum formula* in a derivation is a formula occurrence that is both the conclusion of an introduction rule and the major premiss of an elimination rule. Maximum formulas can be removed by applying *proper reductions*. The rules for proper reductions are as follows. We consider each case in turn.

$(\wedge I)$ followed by $(\wedge E1)$ (analogously in the case involving $(\wedge E2)$)

$$\cfrac{\cfrac{\begin{array}{c}\vdots\,\pi_1\\ a : \phi\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ a : \psi\end{array}}{a : (\phi \wedge \psi)}}{a : \phi} \quad\rightsquigarrow\quad \begin{array}{c}\vdots\,\pi_1\\ a : \phi\end{array}$$

$(\to I)$ followed by $(\to E)$

$$\cfrac{\cfrac{\begin{array}{c}[a : \phi]\\ \vdots\,\pi_1\\ a : \psi\end{array}}{a : (\phi \to \psi)} \qquad \begin{array}{c}\vdots\,\pi_2\\ a : \phi\end{array}}{a : \psi} \quad\rightsquigarrow\quad \begin{array}{c}\vdots\,\pi_2\\ a : \phi\\ \vdots\,\pi_1\\ a : \psi\end{array}$$

$(: I)$ followed by $(: E)$

$$\cfrac{\cfrac{\begin{array}{c}\vdots\,\pi\\ a : \phi\end{array}}{c : a : \phi}}{a : \phi} \quad\rightsquigarrow\quad \begin{array}{c}\vdots\,\pi\\ a : \phi\end{array}$$

$(\Box_i I)$ followed by $(\Box_i E)$

$$\dfrac{\dfrac{\begin{matrix}[a:\Diamond_i c]\\ \vdots\,\pi_1\\ c:\phi\end{matrix}}{a:\Box_i\phi} \qquad \begin{matrix}\vdots\,\pi_2\\ a:\Diamond_i e\end{matrix}}{e:\phi} \quad\rightsquigarrow\quad \begin{matrix}\vdots\,\pi_2\\ a:\Diamond_i e\\ \vdots\,\pi_1[e/c]\\ e:\phi\end{matrix}$$

$(\downarrow I)$ followed by $(\downarrow E)$

$$\dfrac{\dfrac{\begin{matrix}[a:c]\\ \vdots\,\pi_1\\ c:\phi[c/b]\end{matrix}}{a:\downarrow b\phi} \qquad \begin{matrix}\vdots\,\pi_2\\ a:e\end{matrix}}{e:\phi[e/b]} \quad\rightsquigarrow\quad \begin{matrix}\vdots\,\pi_2\\ a:e\\ \vdots\,\pi_1[e/c]\\ e:\phi[e/b]\end{matrix}$$

$(\forall I)$ followed by $(\forall E)$

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\pi\\ a:\phi[c/b]\end{matrix}}{a:\forall b\phi}}{a:\phi[e/b]} \quad\rightsquigarrow\quad \begin{matrix}\vdots\,\pi[e/c]\\ a:\phi[e/b]\end{matrix}$$

We also need reduction rules in connection with the (R_θ) inference rules. A *permutable formula* in a derivation is a formula occurrence that is both the conclusion of a (R_θ) rule and the major premiss of an elimination rule. Permutable formulas in a derivation can be removed by applying *permutative reductions*. The rule for permutative reductions is as follows in the case where the elimination rule has two premisses.

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\tau_1\\ s_1\end{matrix}\ \ \ldots\ \ \begin{matrix}\vdots\,\tau_n\\ s_n\end{matrix} \qquad \begin{matrix}[s_{11}]\ldots[s_{1n_1}]\\ \vdots\,\pi_1\\ \phi\end{matrix} \quad \ldots \quad \begin{matrix}[s_{m1}]\ldots[s_{mn_m}]\\ \vdots\,\pi_m\\ \phi\end{matrix}}{\phi} \qquad \begin{matrix}\vdots\,\pi\\ \theta\end{matrix}}{\psi}$$

$$\rightsquigarrow$$

$$\dfrac{\begin{matrix}\vdots\,\tau_1\\ s_1\end{matrix}\ \ \ldots\ \ \begin{matrix}\vdots\,\tau_n\\ s_n\end{matrix} \qquad \dfrac{\begin{matrix}[s_{11}[\overline{b}/\overline{c}]]\ldots[s_{1n_1}[\overline{b}/\overline{c}]]\\ \vdots\,\pi_1[\overline{b}/\overline{c}]\\ \phi\end{matrix} \qquad \begin{matrix}\vdots\,\pi\\ \theta\end{matrix}}{\psi} \quad \ldots \quad \dfrac{\begin{matrix}[s_{m1}[\overline{b}/\overline{c}]]\ldots[s_{mn_m}[\overline{b}/\overline{c}]]\\ \vdots\,\pi_m[\overline{b}/\overline{c}]\\ \phi\end{matrix} \qquad \begin{matrix}\vdots\,\pi\\ \theta\end{matrix}}{\psi}}{\psi}$$

The nominals in the list $\overline{b}$ are pairwise distinct and new. Note that the side-condition on the rules (R_θ), cf. Figure 3, ensures that the formula $\phi[\overline{b}/\overline{c}]$ is identical to ϕ. This remark also applies to the undischarged assumptions in the derivations of ϕ. The case where the elimination rule has only one premiss is obtained by deleting all instances of the derivation π from the reduction rule above.

A derivation is *normal* if it contains no maximum or permutable formula. In what follows we shall prove a normalization theorem which says that any derivation can be rewritten to a normal derivation by repeated applications of reductions. To this end we need a number of definitions and lemmas.

DEFINITION 5.1
Let $l \in \{1, \ldots, m\}$. The $\Box_l$*-graph* of a derivation π is the binary relation on the set of formula occurrences of π, which is defined as follows. A pair of formula occurrences (ϕ, ψ) is an element of the $\Box_l$-graph of π if and only if it satisfies one of the following conditions.

1. ϕ is of the form $a : \Box_l \neg c$, ψ is of the form $a : \Diamond_l e$, and ϕ is either the major premiss of an instance of $(\Box_l E)$ which has ψ as the relational premiss or ϕ is the minor premiss of an instance of $(\rightarrow E)$ which has ψ as the major premiss.
2. ϕ is of the form $a : \Diamond_l e$, ψ is of the form $a : \Box_l \neg c$, and ϕ is either an assumption discharged at an instance of $(\Box_l I)$ which has ψ as the conclusion or ϕ is the conclusion of an instance of $(\rightarrow I)$ at which ψ is discharged.
3. ϕ and ψ are both of the form $a : \Box_l \neg c$ and ϕ is a non-relational premiss of a (R_θ) rule which has ψ as the conclusion.
4. ϕ and ψ are both of the form $a : \Diamond_l c$ and ψ is a non-relational premiss of a (R_θ) rule which has ϕ as the conclusion.

(Recall that the formulas $a : \Diamond_l e$ and $a : (\Box_l \neg e \rightarrow \bot)$ are identical.) Note that the $\Box_l$-graph of π is a relation on the set of formula occurrences of π; not the set of formulas occurring in π. Also, note that it follows from the definition above that every formula occurrence in a $\Box_l$-graph is of the form $a : \Box_l \neg c$ or $a : \Diamond_l c$.

LEMMA 5.2
The $\Box_l$-graph of a derivation π does not contain cycles.

PROOF. Induction on the structure of π. There are four cases to check according to the definition of a $\Box_l$-graph. The first case has a subcase for each of the rules $(\Box_l E)$ and $(\rightarrow E)$. We consider the first subcase where π has the form

$$\dfrac{\begin{matrix}\vdots\,\tau \\ a : \Box_l \neg c\end{matrix} \qquad \begin{matrix}\vdots\,\sigma \\ a : \Diamond_l e\end{matrix}}{e : \neg c}\,(\Box_l E)$$

Now, the $\Box_l$-graph of π is the union of the $\Box_l$-graph of τ, and the $\Box_l$-graph of σ, and

$$\{(a : \Box_l \neg c, a : \Diamond_l e)\}.$$

By induction the $\Box_l$-graphs of τ and σ do not contain cycles. If the $\Box_l$-graph of π has a cycle, then it contains both of the formula occurrences $a : \Box_l \neg c$ and $a : \Diamond_l e$ indicated above since the $\Box_l$-graphs of τ and σ do not have common nodes. But this cannot be the case, again since the $\Box_l$-graphs of τ and σ do not have common nodes. The second subcase of the first case as well as the second, third, and fourth cases are similar. ∎

DEFINITION 5.3
Let $l \in \{1, \ldots, m\}$. A $\Box_l$*-stubborn formula* in a derivation π is a maximum or permutable formula of the form $a : \Box_l \neg c$ or $a : \Diamond_l c$ and the *potential* of a $\Box_l$-stubborn formula in π is the maximal length of a chain in the $\Box_l$-graph of π that contains the $\Box_l$-stubborn formula. A maximum or permutable formula in π is *stubborn* if it is $\Box_l$-stubborn for some $l \in \{1, \ldots, m\}$.

Note that the notion of potential in the definition above is well-defined since Lemma 5.2 implies that the set of lengths of chains in the $\Box_l$-graph of π is bounded.

LEMMA 5.4
Let π be a derivation where all $\Box_l$-stubborn formulas have potential less than or equal to d. Assume that ϕ is a $\Box_l$-stubborn formula with potential d such that no formula occurrence above a minor or relational premiss of the rule instance of which ϕ is major premiss is $\Box_l$-stubborn and with potential d. Let π' be the derivation obtained by applying the appropriate reduction such that ϕ is removed. Then all $\Box_l$-stubborn formulas in π' have potential less than or equal to d, and moreover, the number of $\Box_l$-stubborn formulas with potential d in π' is less than the the number of $\Box_l$-stubborn formulas with potential d in π.

PROOF. There are four cases to check: if ϕ is a maximum formula, then it is either the conclusion of a $(\Box_l I)$ rule and the major premiss of a $(\Box_l E)$ rule or it is the conclusion of a $(\rightarrow I)$ rule and the major premiss of a $(\rightarrow E)$ rule. If ϕ is a permutable formula, then it is the conclusion of a (R_θ) rule and the major premiss of either a $(\Box_l E)$ rule or a $(\rightarrow E)$ rule. We consider the first case where π and π' have the forms below.

$$\dfrac{\dfrac{\begin{array}{c}[a : \Diamond_l d] \\ \vdots\, \pi_1 \\ d : \neg c\end{array}}{a : \Box_l \neg c}\,(\Box_l I) \qquad \begin{array}{c}\vdots\, \pi_2 \\ a : \Diamond_l e\end{array}}{\begin{array}{c}e : \neg c \\ \vdots\, \tau \\ \psi\end{array}}\,(\Box_l E) \qquad\qquad \begin{array}{c}\vdots\, \pi_2 \\ a : \Diamond_i e \\ \vdots\, \pi_1[e/d] \\ e : \neg c \\ \vdots\, \tau \\ \psi\end{array}$$

Note that any formula occurrence in π' except the indicated occurrences of $a : \Diamond_i e$ and $e : \neg c$ in an obvious way can be mapped to a formula occurrence in π. Let f be the map thus defined (note that f need not be injective as the instance of $(\Box_l I)$ in π might discharge more than one occurrence of $a : \Diamond_l d$). Using the map f, a map from the $\Box_l$-graph of π' to the $\Box_l$-graph of π is defined as follows. An element (ξ, χ) of the $\Box_l$-graph of π' where the formula occurrences ξ and χ both are in the domain of f is mapped to $(f(\xi), f(\chi))$ which straightforwardly can be shown to be an

π' that

does noty f the indicated occurrencesa40.9(of)]TJ /T4 1 Tf 0.12 0 0 -0.12 308.1766 227.1 Tm ( )Tj /T5 1 Tf 70 0 TD
197.9935 191.22 Tm ( )Tj /T5 1 Tf 70.9999 0 TD ( )Tj /T8 1 Tf 51 0 TD ( )Tj /T11 1 Tf 60.054 12.0001 TD ( )Tj /T4 1 Tf 26 -12.0001

$\Box_l$-graph π

greater length contains mentioned formula occurrences. The lemma
followsa40.9(straightforw)12(ardly)60.2(.)-252.98The other threea52.98are similar. ∎

e now readyve the normalization theorem.

THEOREM 5.5 (Normalization)
Any derivation in $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ can be rewritten to a normal derivation by repeated applications 9f proper and permutative reductions.

PROOF. The first step of the theorem is to prove that any derivation can be rewritten to a derivation in which each maximum or permutable formula is stubborn. This is done using a variation of a standard technique (originally given in [19]).

The second step of the theorem is to prove that any derivation which is the result of the first step can be rewritten to a derivation in which all maximum or permutable formulas are of the form $a : \neg c$ (thus, all stubborn formulas have been removed). To any derivation π in which each non-stubborn maximum or permutable formula is of the form $a : \neg c$, we assign the pair (d, k) of non-negative integers where d is the maximal potential of a $\Box_1$-stubborn formula in π or 0 if there is no such formula occurrence and k is the number of $\Box_1$-stubborn formulas in π with potential d. The proof is by induction on such pairs equipped with the lexicographic order. Let π be a derivation to which a pair (d, k) is assigned such that $d > 0$. It is straightforward that there exists a $\Box_1$-stubborn formula ϕ with potential d such that no formula occurrence above a minor or relational premiss of the rule instance of which ϕ is major premiss is $\Box_1$-stubborn and with potential d. (Consider an arbitrary formula occurrence θ in the set of $\Box_1$-stubborn formulas with potential d. We are done if θ satisfies the mentioned criteria. Otherwise, consider instead a $\Box_1$-stubborn formula with potential d such that it is above a minor or relational premiss of the rule instance of which θ is major premiss. This step is repeated until a formula occurrence is found that satisfies the mentioned criteria.) Let π' be the derivation obtained by applying the appropriate reduction such that ϕ is removed. Then by inspecting the reduction rules it is trivial to check that each maximum or permutable formula in π' either is of the form $a : \neg c$ or is stubborn, and moreover, by Lemma 5.4 the pair (d', k') assigned to π' is less than (d, k) in the lexicographic order. Thus, by induction we obtain a derivation in which each maximum or permutable formula either is of the form $a : \neg c$ or for some $l \in \{2, \ldots, m\}$ is $\Box_l$-stubborn. The second step is carried out m times in total whereby a derivation in which each maximum or permutable formula is of the form $a : \neg c$ is obtained.

The third step of the theorem is to prove that any derivation which is the result of the second step can be rewritten to a normal derivation. This is done using the standard technique mentioned in the first step of the theorem. ■

Note that our notion of normalization involves permutative reductions, which is unusual for a classical natural deduction system. Intuitionistic systems, on the other hand, generally involve permutative reductions in connection with inference rules for the connectives $\bot$, $\vee$, and $\exists$.

5.1 The form of normal derivations

Below we shall adapt an important definition from [19] to hybrid logic.

DEFINITION 5.6

A *branch* in a derivation π is a non-empty list $\phi_1, \ldots, \phi_n$ of formula occurrences in π with the following properties.

1. For each $\quad < n$, ϕ_i stands immedialety above ϕ_{i+1}.
2. ϕ_1 is an assumption, or a relational conclusion, or the conclusion af a (R_θ) rule with zero non-relational premisses.
3. ϕ_n is either the end-formula of π or a minor or relational premiss.
4. For each $\quad < n$, ϕ_i is not a minor or relational premiss.

Note that ϕ_1 in the definition above might be a discharged assumption.

LEMMA 5.7

Any formula occurrence in a derivation π belongs to some branch in π.

PROOF. Induction on the structure of π. ■

The definition of a branch leads us to the lemma below which says that a branch in a normal derivation can be split up into three parts: an analytical part in which formulas are broken down in their components by successive applications of the elimination rules, a minimum part in which an instance of the rule $(\perp 1)$ may occur, and a synthetical part in which formulas are put together by by successive applications of the introduction rules. See [20].

LEMMA 5.8
Let $\beta = \phi_1, \ldots, \phi_n$ be a branch in a normal derivation. Then there exists a formula occurrence ϕ_i in β, called the *minimum formula* in β, such that

1. for each $j <$, ϕ_j is the major premiss of an elimination rule, or the non-relational premiss of an instance of $(Nom1)$, or the premiss of an instance of the rule $(\perp 2)$, or a non-relational premiss of an instance of a (R_θ) rule;
2. if $\neq n$, then ϕ_i is a premiss of an introduction rule or the premiss of an instance of the rule $(\perp 1)$; and
3. for each j, where $< j < n$, ϕ_j is a premiss of an introduction rule, or the non-relational premiss of an instance of $(Nom1)$, or a non-relational premiss of an instance of a (R_θ) rule.

PROOF. Let ϕ_i be the first formula occurrence in β which is not the major premiss of an elimination rule, and is not the non-relational premiss of an instance of $(Nom1)$, and is not the premiss of an instance of the rule $(\perp 2)$, and is not a non-relational premiss of an instance of a (R_θ) rule (such a formula occurrence exists in β as ϕ_n satisfies the mentioned criterias). We are done if $= n$. Otherwise ϕ_i is a premiss of an introduction rule or the premiss of an instance of the rule $(\perp 1)$ (by inspection of the rules and the definition of a branch). If ϕ_i is the premiss of an instance of the rule $(\perp 1)$, then ϕ_{i+1} has the form $a : \psi$ where ψ is a propositional symbol. Therefore each ϕ_j, where $< j < n$, is a premiss of an introduction rule, or the non-relational premiss of an instance of $(Nom1)$, or a non-relational premiss of an instance of a (R_θ) rule (by inspection of the rules, the definition of a branch, and normality of π). Similarly, if ϕ_i is a premiss of an introduction rule, then each ϕ_j, where $< j < n$, is a premiss of an introduction rule or a non-relational premiss of an instance of a (R_θ) rule. ■

The lemma above is more technically involved than the corresponding result in [19], the reason being the disturbing effect of $(Nom1)$, $(\perp 2)$, and the (R_θ) rules. In the theorem below we make use of the following definition.

DEFINITION 5.9
The notion af a *subformula* is defined by the conventions that

- ϕ is a subformula of ϕ;
- if $\psi \wedge \theta$ or $\psi \rightarrow \theta$ is a subformula of ϕ, then so are ψ and θ;
- if $a : \psi$ or $\Box_i \psi$ is a subformula of ϕ, then so is ψ; and
- if $\downarrow a \psi$ or $\forall a \psi$ is a subformula of ϕ, then so is $\psi[c/a]$.

A formula $a : \phi$ is a *quasi-subformula* of a formula $c : \psi$ if and only if ϕ is a subformula of ψ.

Now the theorem which says that normal derivations satisfy a version of the subformula property.

THEOREM 5.10 (Quasi-subformula property)
Let π be a normal derivation of ϕ from Γ in $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$. Moreover, let θ be a formula occurrence in π such that

1. θ is not an assumption discharged by an instance of the rule $(\bot 1)$ where the discharged assumption is the major premiss of an instance of $(\rightarrow E)$;

2. θ is not an occurrence of $a : \bot$ in a branch whose first formula is an assumption discharged by an instance of the rule $(\bot 1)$ where the discharged assumption is the major premiss of an instance of $(\rightarrow E)$; and

3. θ is not an occurrence of $a : \bot$ in a branch whose first formula is the conclusion of a (R_θ) rule with zero non-relational premisses.

Then θ is a quasi-subformula of ϕ, or of some formula in Γ, or of some relational premiss, or of some relational conclusion, or of some relationally discharged assumption.

PROOF. First a small convention: the *order* of a branch in π is the number of formula occurrences in π which stand below the last formula occurrence of the branch. Now consider a branch $\beta = \phi_1, \ldots, \phi_n$ in π of order p. By induction we can assume that the theorem holds for all formula occurrences in branches of order less than p. Note that it follows from Lemma 5.8 that any formula occurrence ϕ_j such that $j \leq$, where ϕ_i minimum formula in β, is a quasi-subformula of ϕ_1, and similarly, any ϕ_j such that $j >$ is a quasi-subformula of ϕ_n.

We first consider ϕ_n. We are done if ϕ_n is the end-formula ϕ or a relational premiss. Otherwise ϕ_n is the minor premiss of an instance of $(\rightarrow E)$. If the major premiss of this instance of $(\rightarrow E)$ is not an assumption discharged by an instance of the rule $(\bot 1)$, then we are done by induction as the major premiss belongs to a branch of order less than p. If the major premiss of the instance of $(\rightarrow E)$ in question is an assumption discharged by an instance of the rule $(\bot 1)$, then we are done by induction as the conclusion of this instance of $(\bot 1)$, which has the same form as ϕ_n, belongs to a branch of order less than p.

We now consider ϕ_1. We are done if ϕ_1 is an undischarged assumption, or a relationally discharged assumption, or a relational conclusion. If ϕ_1 is the conclusion of a (R_θ) rule with zero non-relational premisses, and if ϕ_1 is not of the form $a : \bot$, then ϕ_1 has the same form as the minimum formula which is a premiss of an introduction rule and hence quasi-subformula of ϕ_n. If ϕ_1 is not discharged by an instance of $(\bot 1)$, then it is discharged by an instance of $(\rightarrow I)$ with a conclusion that belongs to β or to some branch of order less than p. If ϕ_1 is discharged by an instance of $(\bot 1)$, then we have three cases. We are done if $n = 1$. If $n \neq 1$ and ϕ_1 is the minimum formula of β, then ϕ_1 is a premiss of an introduction rule and hence quasi-subformula of ϕ_n. If $n \neq 1$, but ϕ_1 is not the minimum formula of β, then ϕ_1 is either the major premiss of an instance of $(\rightarrow E)$ or a non-relational premiss of an instance of a (R_θ) rule. The first case is clear and in the second case ϕ_1 has the same form as the minimum formula which is a premiss of an introduction rule and hence quasi-subformula of ϕ_n. ∎

The first two exceptions in the theorem above are inherited from the standard natural deduction system for classical logic, see [19], whereas the third one is related to the possibility of having a (R_θ) rule with zero non-relational premisses.

REMARK.
If the formula occurrence θ is not covered by one of the three exceptions, then it is a quasi-subformula of ϕ, or of some formula in Γ, or of a formula of the form $a : c$ or $a : \Diamond_i c$ (since relational premisses, relational conclusions, and relationally discharged assumptions are of the form $a : c$ or $a : \Diamond_i c$). Note that the formulation of the theorem involves the notion of a branch in what appears to be an indispensable way.

$$\frac{}{\Gamma, \phi \vdash \Delta, \phi}\,(Axiom) \qquad \frac{}{\Gamma, a : \bot \vdash \Delta}\,(\bot)$$

$$\frac{a : \phi, a : \psi, \Gamma \vdash \Delta}{a : (\phi \wedge \psi), \Gamma \vdash \Delta}\,(\wedge L) \qquad \frac{\Gamma \vdash \Delta, a : \phi \quad \Gamma \vdash \Delta, a : \psi}{\Gamma \vdash \Delta, a : (\phi \wedge \psi)}\,(\wedge R)$$

$$\frac{\Gamma \vdash \Delta, a : \phi \quad a : \psi, \Gamma \vdash \Delta}{a : (\phi \rightarrow \psi), \Gamma \vdash \Delta}\,(\rightarrow L) \qquad \frac{a : \phi, \Gamma \vdash \Delta, a : \psi}{\Gamma \vdash \Delta, a : (\phi \rightarrow \psi)}\,(\rightarrow R)$$

$$\frac{\Gamma \vdash \Delta, a : \Diamond_i e \quad e : \phi, \Gamma \vdash \Delta}{a : \Box_i \phi, \Gamma \vdash \Delta}\,(\Box_i L) \qquad \frac{a : \Diamond_i c, \Gamma \vdash \Delta, c : \phi}{\Gamma \vdash \Delta, a : \Box_i \phi}\,(\Box_i R)^*$$

$$\frac{a : \phi, \Gamma \vdash \Delta}{c : a : \phi, \Gamma \vdash \Delta}\,(: L) \qquad \frac{\Gamma \vdash \Delta, a : \phi}{\Gamma \vdash \Delta, c : a : \phi}\,(: R)$$

$$\frac{\Gamma \vdash \Delta, a : e \quad e : \phi[e/b], \Gamma \vdash \Delta}{a : \downarrow b \phi, \Gamma \vdash \Delta}\,(\downarrow L) \qquad \frac{a : c, \Gamma \vdash \Delta, c : \phi[c/b]}{\Gamma \vdash \Delta, a : \downarrow b \phi}\,(\downarrow R)^*$$

$$\frac{b : \phi[e/a], \Gamma \vdash \Delta}{b : \forall a \phi, \Gamma \vdash \Delta}\,(\forall L) \qquad \frac{\Gamma \vdash \Delta, b : \phi[c/a]}{\Gamma \vdash \Delta, b : \forall a \phi}\,(\forall R)^*$$

$*$ c does not occur free in the conclusion.

FIGURE 6. Gentzen rules for connectives

$$\frac{a : a, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\,(Ref) \qquad \frac{\Gamma \vdash \Delta, a : c \quad \Gamma \vdash \Delta, a : \phi}{\Gamma \vdash \Delta, c : \phi}\,(Nom1)^*$$

$$\frac{\Gamma \vdash \Delta, a : c \quad \Gamma \vdash \Delta, a : \Diamond_i b \quad c : \Diamond_i b, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\,(Nom2)$$

$*$ ϕ is a propositional symbol (ordinary or a nominal).

FIGURE 7. Gentzen rules for nominals

6 Natural deduction at work

We shall in this section consider a cut-free Gentzen system corresponding to our natural deduction system $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$. Gentzen systems are like natural deduction systems characterized by having two different kinds of rules for each non-nullary connective, but whereas natural deduction rules either introduce or eliminate a connective, Gentzen rules either introduce a connective on the left-hand side of a sequent or introduce the connective on the right-hand side of a sequent. Gentzen rules for hybrid logic are given in Figures 6, 7, and 8. All formulas in the rules are satisfaction statements. By convention Γ, ϕ and ϕ, Γ are abbreviations for $\Gamma \cup \{\phi\}$, and similarly, Γ, Δ is an abbreviation for $\Gamma \cup \Delta$. All rules with a name of the form $(\ldots R)$ introduce a connective on the

$$\frac{\Gamma \vdash \Delta, s_1 \quad \dots \quad \Gamma \vdash \Delta, s_n \quad s_{11}, \dots, s_{1n_1}, \Gamma \vdash \Delta \quad \dots \quad s_{m1}, \dots, s_{mn_m}, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \, (R_\theta)^*$$

$*$ None of the nominals in $\bar{c}$ occur free in Γ or Δ.

FIGURE 8. Gentzen rules for geometric theories

right-hand side of a sequent, and similarly, all rules with a name of the form $(\dots L)$ introduce a connective on the left-hand side. We assume that we are working with a fixed basic geometric theory $\mathbf{T}$. The Gentzen system for $\mathcal{H}(\mathcal{O})$ will be denoted $\mathbf{G}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$. Note that a derivation of a sequent $\Gamma \vdash \Delta$ in $\mathbf{G}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ has the property that every formula occurring in the derivation either is a quasi-subformula of a formula in Γ or Δ or is a quasi-subformula of a formula of the form $a : c$ or $a : \Diamond_i c$. Normal natural deduction derivations have an analogous property, cf. the remark following Theorem 5.10. Of course, this property is lost if we add the cut rule

$$\frac{\Gamma \vdash \Delta, \phi \qquad \phi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \, (Cut)$$

to the system $\mathbf{G}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$.

We now use the normalization theorem for our natural deduction system $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ to prove a lemma which implies the completeness of $\mathbf{G}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$.

LEMMA 6.1
Let π be a normal derivation of ψ from Γ in $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$, and moreover, let $\{a_1 : \neg\phi_1, \dots, a_n : \neg\phi_n\} \subseteq \Gamma$ where $n \geq 0$, let $\Gamma^* = \Gamma - \{a_1 : \neg\phi_1, \dots, a_n : \neg\phi_n\}$, and let $\Delta = \{a_1 : \phi_1, \dots, a_n : \phi_n\}$. Then there exists a derivation of the sequent $\Gamma^* \vdash \Delta, \psi$ in $\mathbf{G}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$.

PROOF. We first prove that the lemma holds for the Gentzen system $\mathbf{G}'_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ which is obtained from the system $\mathbf{G}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ by replacing the axiom $(\bot)$ by the rule

$$\frac{\Gamma \vdash \Delta, a : \bot}{\Gamma \vdash \Delta, \Delta'}$$

Observe that a derivation τ in $\mathbf{G}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ ($\mathbf{G}'_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$) of a sequent $\Gamma \vdash \Delta$ can be transformed into a derivation in $\mathbf{G}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ ($\mathbf{G}'_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$) of any sequent $\Gamma \cup \Gamma' \vdash \Delta \cup \Delta'$ simply by adding Γ' and Δ' to the sets of formulas in the sequents of τ and by renaming of nominals.

The proof that the lemma holds for $\mathbf{G}'_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ is by induction on the number of rule instances in π. We only cover the case where ψ is the conclusion of an elimination rule; the other cases are straightforward. Let $\beta = \psi_1, \dots, \psi_n$ be a branch in π such that $\psi_n = \psi$. Since ψ_n is the conclusion of an elimination rule, each formula occurrence in β except ψ_n is the major premiss of an elimination rule, cf. Lemma 5.8. Thus, the formula occurrence ϕ_1 cannot be a discharged assumption. So ϕ_1 is either an undischarged assumption, or a relational conclusion, or the conclusion af a (R_θ) rule with zero non-relational premisses. The cases where ϕ_1 is a relational conclusion or the conclusion af a (R_θ) rule with zero non-relational premisses are straightforward. If ϕ_1 is an undischarged assumption, then we split up in subcases depending on the form of ϕ_1. Note that $\phi_1 \in \Gamma$. We only cover the subcase where ϕ_1 is of the form $a : \Box_i \phi$; the other subcases are similar. So we have a derivation of $a : \Diamond_i e$ from Γ for some nominal e, and moreover, we have a derivation of ψ from $e : \phi, \Gamma$. By induction we get derivations in $\mathbf{G}'_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ of the sequents $\Gamma^* \vdash \Delta, a : \Diamond_i e$ and

$e : \phi, \Gamma^* \vdash \Delta, \psi$. It is then easy to build a derivation in $\mathbf{G'}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ of $\Gamma^* \vdash \Delta, \psi$ using the rule $(\Box_i L)$ (note that $a : \Box_i \phi, \Gamma^* = \Gamma^*$ since $a : \Box_i \phi = \phi_1$ and $\phi_1 \in \Gamma$).

It is straightforward to show that a derivation τ in $\mathbf{G}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ of a sequent $\Gamma \vdash \Delta, a : \bot$ can be transformed into a derivation in $\mathbf{G}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ of the sequent $\Gamma \vdash \Delta$ by removing $a : \bot$ from the right-hand side sets of formulas in the sequents of τ and by replacing instances of $(Axiom)$ by instances of $(\bot)$. It follows that the lemma holds for $\mathbf{G}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$. ■

Now soundness and completeness.

THEOREM 6.2 (Soundness and completeness)
The two statements below are equivalent.

1. $\Gamma \vdash \Delta$ is derivable in $\mathbf{G}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$.
2. For any $\mathbf{T}$-model $\mathcal{M}$ and any assignment g, if, for any formula $\theta \in \Gamma$, $\mathcal{M}, g \models \theta$, then for some formula $\psi \in \Delta$, $\mathcal{M}, g \models \psi$.

PROOF. Soundness is by induction on the structure of the derivation of $\Gamma \vdash \Delta$. Completeness follows from Lemma 6.1. ■

7 Comparison to related work

Our natural deduction systems share several features with the hybrid-logical tableau and Gentzen systems given in [3] and the tableau system given in [5], for example the feature that all formulas in derivations are satisfaction statements. However, since our systems are in natural deduction style, we provide a proof-theoretic analysis in the form a normalization theorem and a theorem which says that any normal derivation satisfies a version of the subformula property, namely the quasi-subformula property. On the other hand, tableau systems and cut-free or analytic Gentzen systems in general trivially satisfy the subformula property. Indeed, if the quasi-subformula property is formulated as appropriate for the tableau system given in [3], then it is trivial to check that it has this property. A similar remark applies to the Gentzen system of [3]. Another difference between our work and [3] is that we consider additional inference rules corresponding to first-order conditions expressed by geometric theories whereas [3] considers tableau systems extended with axioms that are so-called pure hybrid-logical formulas, that is, formulas that contain no ordinary propositional symbols (thus, the only propositional symbols in such formulas are nominals). Note in this connection that a first-order condition expressible by a geometric theory is not necessarily expressible by a pure formula of hybrid logic unless the $\forall$ binder is used.

The use of geometric theories in the context of proof-theory traces back to [25] where it was pointed out that formulas in basic geometric theories correspond to simple natural deduction rules for intuitionistic modal logic. First-order conditions expressed by geometric theories cover a very wide class of logics. This is, for example, witnessed by the fact that any so-called *Geach axiom schema*, that is, modal-logical axiom schema of the form

$$\Diamond^k \Box^m \phi \rightarrow \Box^l \Diamond^n \phi$$

where $\Box^j$ (respectively $\Diamond^j$) is an abbreviation for a sequence of j occurrences of $\Box$ (respectively $\Diamond$), corresponds to a formula of the form required in a basic geometric theory. To be precise, such a Geach axiom schema corresponds to the first-order formula

$$\forall a \forall b \forall c((R^k(a,b) \wedge R^l(a,c)) \rightarrow \exists d(R^m(b,d) \wedge R^n(c,d)))$$

where $R^0(a, b)$ means $a = b$ and $R^{j+1}(a, b)$ means $\exists e(R(a, e) \wedge R^j(e, b))$. The displayed formula is then equivalent to a formula of the form required in a basic geometric theory. See [25, 2]. In the natural deduction system considered in [25], a distinction is made between the language of ordinary modal logic and a metalanguage involving atomic first-order formulas of the form $R(a, c)$ together with formulas of the form $a : \phi$ where ϕ is a formula of ordinary modal logic. One contribution of the present paper is to point out that basic geometric theories correspond to natural deduction rules for hybrid logic where no such distinction between an object language and a metalanguage is made. It should be mentioned that a natural deduction system for classical modal logic which is similar to the system of [25] has been given in [2]. However, one difference is that only Horn clause theories are considered in the latter work.

The Gentzen systems we consider in this paper are somewhat similar to a Gentzen system given in [3]. One difference, however, is that in [3] a different set of rules for equality (in connection with nominals) is used.

The feature of our systems that all formulas in derivations are satisfaction statements is at a general level in line with the fundamental idea of [14, 15] which is to prefix formulas in derivations by metalinguistic indexes, or labels, with the aim of regulating the proof process. In fact, the labelled deductive systems of [15] are proposed as a systematic way of giving proof systems to many different logics. Note that the work of [25] fits naturally into this framework. It should also be mentioned that labelled deductive systems are the basis for the natural deduction systems for substructural logics given in [12]. The crucial difference between the work of [14, 15, 25, 12] and our work is that the indexes, or labels, used in the mentioned work belong to a metalanguage whereas in our systems they are part of the object language, namely the language of hybrid logic. Thus, in the the terminology of [3], the metalanguage has been internalized in the object language.

Also the paper [23] should be mentioned here. In that paper a natural deduction system for a logic of situations similar to hybrid logic is given; the system in question is, however, quite different from ours, see [11] for a comparison. See also [24, 27].

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