Sahlqvist theory and transfer results for Hybrid Logic*

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Abstract

In the first part, we prove completeness for Sahlqvist extensions of the basic hybrid logic, and we prove that the basic hybrid logic has the Beth property. We also show that extensions of the basic hybrid logic that contain Sahlqvist formulas as well as pure formulas are in general not complete.

In the second part, we define for every Kripke complete modal logic L its hybrid companion L_H and investigate which properties transfer from L to L_H. For a specific class of logics, we present a satisfiability-preserving translation from L_H to L. We prove that for this class of logics, complexity, (uniform) interpolation, and finite axiomatization transfer from L to L_H. We also provide examples showing that, in general, none of complexity, decidability or the finite model property transfer.

Part I: Sahlqvist theory for hybrid logic

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1 Introduction

Hybrid logic comes with a general completeness result: Every extension with pure axioms of the basic hybrid logic with [name] and [bg] rules is complete [4, 5]. A pure axiom is a formula constructed from nominals only, thus not containing arbitrary proposition letters. Pure axioms correspond to first order frame condition and are quite expressive [4]. For instance, $i \to \neg\Diamond i$ defines the class of irreflexive frames.

We can compare this general result with Sahlqvist's theorem for modal logic, a similar general completeness result. Several questions come to mind. Is every Sahlqvist axiom expressible as a pure axiom? No, the Church–Rosser axiom $\Diamond\Box p \to \Box\Diamond p$ is a counterexample [12]. On the other hand, in the presence of the tense modalities, the answer is yes [16]. This gives us two new questions:

1. Is the extension of the basic hybrid logic with a set of Sahlqvist axioms always complete? That is, does Sahlqvist's theorem go through for hybrid logic?
2. Can we combine the two general completeness results? That is, is every extension of the basic hybrid logic with a set Σ of Sahlqvist axioms and a set Π of pure axioms complete with respect to the class of frames defined by Σ and Π together?

First part of the paper answers both questions. We show that every extension of the basic hybrid logic with canonical modal axioms (and hence, Sahlqvist axioms) is complete even without the [name] and [bg] rules, and we give a Sahlqvist formula σ and a pure formula π such that the basic hybrid logic extended with the axioms σ and π is incomplete even in the presence of the [name] and the [bg] rules.

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As a corollary, we solve an embarrassing open problem in hybrid logic: whether Beth's definability property holds (cf. [9] for a discussion of this open problem and some partial results). Another corollary of our analysis is that [name] and [bg] are superfluous not only for the basic hybrid system, but also for every Sahlqvist extension. This is a desirable result, since these rules are non-orthodox in the sense that they involve syntactic side-conditions, much like the Gabbay's irreflexivity rule.

This part of the paper is organized as follows. This section briefly recalls hybrid logic. Section 2 shows Sahlqvist's theorem for hybrid logic. In Section 3 we derive interpolation and Beth's property. Section 4 shows that a combination of Sahlqvist and pure axioms is not guaranteed to be complete. We conclude in Section 5.

2 Hybrid logic

What follows is a short textbook-style presentation of hybrid logic, following [4]. Hybrid logic is the result of extending the basic modal language with a second sort of atomic propositions called nominals, and with satisfaction operators. The nominals behave similar to ordinary proposition letters, except that their interpretation in models is restricted to singleton sets. In other words, nominals act as names for worlds. Satisfaction operators allow one to express that a formula holds at the world named by nominal. For example, $@_i p$ expresses that p holds at the world named by the nominal i.

Formally, let PROP be a countably infinite set of proposition letters and NOM a countably infinite set of nominals.[1] The formulas of the basic hybrid logic are given as follows.

$$\phi ::= p \mid i \mid \top \mid \neg\phi \mid \phi \wedge \psi \mid \Diamond\phi \mid @_i\phi$$

where $p \in$ PROP and $i \in$ NOM. The truth definition for the nominals is the same as for the proposition letters: our models are of the form $M = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is a frame and V a valuation function for the proposition letters and nominals. The truth definition for the nominals is the same as for the proposition letters: $M, w \models i$ iff $w \in V(i)$. The only difference is in the admissible valuations: only valuation functions are allowed that assign to each nominal a singleton set. The interpretation of the satisfaction operators is as could be expected: $M, w \models @_i\phi$ iff $M, v \models \phi$, where $V(i) = \{v\}$.

Next, let us turn to axiomatizations for this language. Let Δ be the set of axioms given by [agree], [back], [introduction], [ref], and [self-dual], for all $i, j \in$ NOM.

[agree]	$@_j @_i p \to @_i p$
[back]	$\Diamond @_i p \to @_i p$
[introduction]	$i \wedge p \to @_i p$
[ref]	$@_i i$
[self-dual]	$@_i p \leftrightarrow \neg @_i \neg p.$

Let $\mathbf{K}_{\mathcal{H}(@)}$ be the smallest set of formulas containing all tautologies, axioms $\Box(p \to q) \to (\Box p \to \Box q)$, $@_i(p \to q) \to (@_i p \to @_i q)$ for $i \in$ NOM, and the axioms in Δ, closed under *modus ponens*, *uniform substitution of formulas for proposition letters and nominals for nominals*, *generalization* (If $\vdash \phi$ then $\vdash \Box\phi$), and @-*generalization* (If $\vdash \phi$ then $\vdash @_i\phi$). Given a set Σ of hybrid formulas, the logic $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ is obtained by adding the formulas in Σ to $\mathbf{K}_{\mathcal{H}(@)}$ as extra axioms, and closing under the given rules.

Theorem 2.1 $\mathbf{K}_{\mathcal{H}(@)}$ *is strongly sound and complete for the class of all frames.*

A sketch of the proof of Theorem 2.1 can be found in [4]. Theorem 2.1 also follows from our results in Section 3.

[1] The results discussed in this paper and their proofs also apply if NOM is finite.

A general completeness result holds for extensions of the basic logic with pure axioms, provided two extra derivation rules are added to the calculus. A pure axiom is an axiom that contains no proposition letters, only nominals.

[name]	If $\vdash @_i\phi$ and i does not occur in ϕ, then $\vdash \phi$
[bg]	If $\vdash @_i\Diamond j \to @_j\phi$ where $j \neq i$ and j does not occur in ϕ, then $\vdash @_i\Box\phi$

Several variants of these rules occur in literature, under names such as *Cov* [12] and *Name and Paste* [4]. In the above shape, the rules first appear in [5].

Let $\mathbf{K}^+_{\mathcal{H}(@)}$ be the logic obtained by adding these two inference rules to $\mathbf{K}_{\mathcal{H}(@)}$. Given a set Σ of hybrid formulas, the logic $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ is obtained by adding the formulas in Σ to $\mathbf{K}^+_{\mathcal{H}(@)}$ as extra axioms, and closing under all rules, including the two extra rules.

Theorem 2.2 ([5]) *Let Σ be any set of pure formulas. Then $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ is strongly sound and complete for the frame class defined by Σ.*

One question left open by Theorem 2.2 is *when are the rules needed*. While the proof of Theorem 2.2 is based on a Henkin model construction that crucially depends on the presence of the rules, this does not exclude the possibility of completeness without rules. Another question that remains is if every extension of $\mathbf{K}^+_{\mathcal{H}(@)}$ with Sahlqvist axioms is complete. Recall from the introduction that not every Sahlqvist axiom corresponds to a pure axiom.

3 Sahlqvist completeness for hybrid logic

Consider frames of the form $\mathfrak{F} = \langle W, R, (R_i)_{i\in nom}, (S_i)_{i\in\text{NOM}}\rangle$, where each R_i is a binary relation on W and each S_i is a subset of W. Let us call such frames *non-stanfard frames*, to distinguish them from the ordinary frames. Let us say that such a non-standard frame is *nice* if for each $i \in \text{NOM}$, S_i is a singleton and $\forall xy(R_ixy \leftrightarrow S_iy)$. A non-standard model is a pair $(\mathfrak{F}, V)$ where $\mathfrak{F}$ is a non-standard frame and the valuation V interprets the proposition letters (the interpretation of the nominals is already given by $\mathfrak{F}$).

Viewing the satisfaction operators as modalities and the nominals as modal constants, we can evaluate hybrid formulas on non-standard models: we simply extend the usual satisfaction definition for modal logic with the following clauses:

$$\begin{array}{rcl} \mathfrak{M}, w \Vdash i & \text{iff} & w \in S_i, \\ \mathfrak{M}, w \Vdash @_i\phi & \text{iff} & \exists w'(wR_iw' \text{ and } \mathfrak{M}, w' \Vdash \phi). \end{array}$$

By this change in semantics, the formulas in Δ define properties of non-standard frames. For instance, [self-dual] says that the relations R_i are functional. As a matter of fact, each of the axioms, being in Sahlqvist form, is canonical and has a first-order correspondent, given below.

[agree]	$\forall xyz(R_jxy \wedge R_iyz \to R_ixz)$
[back]	$\forall xyz(Rxy \wedge R_iyz \to R_ixz)$
[introduction]	$\forall x(S_ix \to R_ixx)$
[ref]	$\forall x\exists y(R_ixy \wedge S_iy)$
[self-dual]	$\forall xyz(R_ixy \wedge R_ixz \to y = z)$.

It is not hard to see from these first-order correspondents that the following holds.

Lemma 3.1 *A point-generated non-standard frame $\mathfrak{F}$ is* nice *iff $\mathfrak{F} \models \Delta$.*

By canonicity, the following completeness result follows.[2]

Corollary 3.2 *Let Σ be a set of canonical modal formulas. Then $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ is strongly sound and complete for the class of nice non-standard frames validating Σ.*

With each nice non-standard model $\mathfrak{M} = \langle W, R, (R_i)_{i \in \text{NOM}}, (S_i)_{i \in \text{NOM}}, V\rangle$, we can associate a standard hybrid model $\mathfrak{M}^+ = \langle W, R, V \cup \{(i, S_i) \mid i \in \text{NOM}\}\rangle$. In fact, this operation on models is bijective, in the sense that for every standard hybrid model $\mathfrak{M}$, there is exactly one nice non-standard model $\mathfrak{N}$ such that $\mathfrak{M} = \mathfrak{N}^+$. A straightforward inductive argument shows that the operation $(\cdot)^+$ preserves local truth of formulas: for all hybrid formulas ϕ, $\mathfrak{M}, w \models \phi$ iff $\mathfrak{M}^+, w \models \phi$. Moreover, if ϕ contains no nominals or satisfaction operators, then ϕ is valid on the underlying frame of $\mathfrak{M}^+$ iff ϕ is valid on the underlying (non-standard) frame of $\mathfrak{M}^+$. Hence, we obtain the following.

Theorem 3.3 *Let Σ be a set of canonical modal formulas not containing nominals or satisfaction operators. Then $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ is strongly sound and complete for the class of frames defined by Σ.*

Corollary 3.4 *Every extension of $\mathbf{K}_{\mathcal{H}(@)}$ with modal Sahlqvist axioms not containing nominals or satisfaction operators is strongly sound and complete for the class of frames defined by the axioms.*

Gargov and Goranko [12] obtain a similar result for the hybrid language with the global modality, via a slightly different route.

4 Interpolation and Beth's property

An open problem in hybrid logic was whether the basic hybrid logic has Beth's definability property [9]. It is known that interpolation fails [1], but Beth's property is a bit weaker. For Beth's property to follow, we need just a restricted version of interpolation: the interpolant may only contain shared proposition letters but it can contain nominals occurring only in the antecedent or in the consequent. As an immediate corollary of Theorem 1 we obtain this form of interpolation and hence Beth's property.

A logic has *interpolation over proposition letters* if whenever $\phi \to \psi$ is provable, there exists an interpolant θ, such that $\phi \to \theta$ and $\theta \to \psi$ are provable, and all proposition letters occurring in θ occur both in ϕ and in ψ.

Marx [20, Corollary B.4.1] showed that every canonical modal logic which is complete with respect to a universal Horn definable class of frames has interpolation over proposition letters. With the exception of [ref], all first-order correspondents of axioms in Δ are universal Horn sentences. The [ref] is itself a formula without proposition letters. By [22], extending a logic that has interpolation over proposition letters with formulas without proposition letters yields again a logic with interpolation over proposition letters. Hence, we obtain the following.

Theorem 4.1 *Let Σ be a set of canonical modal formulas with universal Horn correspondents. Then $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ has interpolation over proposition letters.*

A modal logic is said to have Beth's definability property if every implicit definition can be made explicit. More concretely, let $\Gamma(p)$ be a set of formulas containing the proposition letter p and possibly other proposition letters and nominals. $\Gamma(p)$ defines p if in all models

[2] An apparent technical problem: the axiomatization of $\mathbf{K}_{\mathcal{H}(@)}$ includes the following K axiom for the satisfaction operators: $\vdash @_i(p \to q) \to @_i p \to @_i q$. This axiom relies on an interpretation of satisfaction operators as boxes. On the other hand, in the present section, we treat satisfaction operators as diamonds. Hence, strictly speaking, we need the dual K-axiom: $\vdash \overline{@_i}(p \to q) \to \overline{@_i}p \to \overline{@_i}q$. This problem is only apparent, since the latter is derivable from the former in the presence of [self-dual] and [ref].

in which both $\Gamma(p)$ and $\Gamma(p')$ are true at every state, also $p \leftrightarrow p'$ is true at every state.[3] In other words, $\Gamma(p)$ defines p if $\Gamma(p) \cup \Gamma(p') \models^{glo} p \leftrightarrow p'$, where $\models^{glo}$ denotes global entailment. Beth's property states that whenever this obtains, there exists a formula θ in which p does not occur, such that $\Gamma(p) \models^{glo} p \leftrightarrow \theta$. Clearly, θ is an explicit definition of p, relative to the theory $\Gamma(p)$.

Beth's property is a completeness theorem for definitions: it states that every semantic definition corresponds to an explicit, syntactic, definition. A standard argument derives the following from Theorem 4.1.

Theorem 4.2 *Let Σ be a set of canonical modal formulas with universal Horn correspondents. Then $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ has Beth's definability property.*

5 Combining pure and Sahlqvist axioms

As we mentioned in the introduction, not every Sahlqvist axiom corresponds to a pure axiom. It is natural to ask if completeness obtains when we extend the basic hybrid logic $\mathbf{K}^+_{\mathcal{H}(@)}$ with a combination of pure and canonical axioms. The answer is negative.

Theorem 5.1 *There is a pure axiom π and a Sahlqvist axiom σ such that the hybrid logic $\mathbf{K}^+_{\mathcal{H}(@)}\{\pi, \sigma\}$ is not complete for the frame class defined by $\pi \wedge \sigma$.*

Proof: Consider the following axioms (the first-order frame conditions they define are given as well):

[cr]	$\Diamond\Box p \to \Box\Diamond p$	$\forall xyz(Rxy \wedge Rxz \to \exists u(Ryu \wedge Rzu))$
[nogrid]	$\Diamond(i \wedge \Diamond j) \to \Box(\Diamond j \to i)$	$\forall xyzu(Rxy \wedge Rxz \wedge Ryu \wedge Rzu \to y = z)$
[func]	$\Diamond p \to \Box p$	$\forall xyz(Rxy \wedge Rxz \to y = z)$

[cr] is a Sahlqvist formula and [nogrid] is pure. As can be easily seen from the first-order correspondents, every frame validating [cr] and [nogrid] validates [func]. However, we claim that [func] is not derivable from [cr] and [nogrid] (not even using the [name] and [bg] rules). To see this, consider ω^ω, i.e., the countably branching tree of infinite depth. Let $\mathfrak{F}$ be the general frame for this structure in which the admissible sets are exactly the finite and co-finite sets [4].

Obviously, every axiom of hybrid logic is valid on $\mathfrak{F}$. Furthermore, the set of formulas valid on $\mathfrak{F}$ is closed under all derivation rules of hybrid logic, including [name] and [bg] (the latter follows from the fact that every singleton set is admissible). As we are about to show, $\mathfrak{F} \models$ [cr], $\mathfrak{F} \models$ [nogrid] and $\mathfrak{F} \not\models$ [func]. Recall that $\mathbf{K}^+_{\mathcal{H}(@)}\{[\text{cr}], [\text{nogrid}]\}$ is defined as the *smallest* such set of formulas, i.e., the smallest set of formulas containing all axioms of hybrid logic, [cr]and [nogrid], that is closed under all inference rules (including [name] and [bg]). It now clearly follows that [func] $\notin \mathbf{K}^+_{\mathcal{H}(@)}\{[\text{cr}], [\text{nogrid}]\}$.

To show that $\mathfrak{F} \models$ [cr], suppose $\mathfrak{F}, V, w \Vdash \Diamond\Box p$. Since $V(p)$ admissible, it must be either finite or co-finite. Since w satisfies $\Diamond\Box p$, there must be a point with only successors satisfying p. Since every point in ω^ω has infinitely many successors, it follows that $V(p)$ must be infinite, hence co-finite. It follows that every world has a successor satisfying p, and therefore, $\mathfrak{F}, V, w \models \Box\Diamond p$. This establishes the validity of [cr]. That $\mathfrak{F} \models$ [nogrid] and $\mathfrak{F} \not\models$ [func] is clear. $\Box$

[3] Here, p' is a proposition letter not occurring in Γ, and $\Gamma(p')$ is the result of replacing all occurrences of p by p' in $\Gamma(p)$.

6 Conclusion

In hybrid logic we have two general completeness results: Sahlqvist's theorem and the theorem for pure axioms. We showed that they cannot be combined, at least not in the obvious way. The situation is radically different in tense hybrid logic. Here the combination problem is not relevant, as every Sahlqvist axiom is expressible as a pure axiom. It seems that for hybrid logic a similar conclusion holds as for modal logics with the difference operator. In both cases, there is no general completeness theorem like Venema's SD theorem [25] except in the case of tense logics. Venema speculated that in the non-tense case, one can always get completeness by adding suitable axioms, but there is no general recipe indicating which axioms. The axiom that needs to be added in the case of Theorem 5.1 to restore completeness is easy to find. In fact, [func] itself suffices.

A slightly different interpretation of Theorem 3.3 can be summarized as follows:

> Strong completeness transfers from modal logics to the corresponding hybrid logics, provided that the logics in question are canonical.

In other words, Theorem 3.3 can be viewed as a (partial) transfer result. The question of transfer from modal logics to their hybrid companions is studies in more detail in Part II of this paper. There, among other things, a similar transfer result for completeness is presented for modal logics that admit filtration. These results complement each other, since the latter applies to non-canonical logics such as **GL** and **Grz**.

Part II: Transfer results for hybrid logic

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1 Introduction

For every Kripke complete modal logic L, we define its hybrid companion L_H as the hybrid logic of the class of frames defined by L. This paper addresses the following question, raised by Gargov and Goranko [12]:

> If a modal logic L has a property $\mathfrak{P}$, does its hybrid companion L_H have $\mathfrak{P}$ too?

Gargov and Goranko ask this question in the context of a hybrid language with the universal modality. They show, for example, that canonicity transfers. In this paper, we will focus on a simpler hybrid language, namely the enrichment of the basic modal language with nominals only. In order to obtain transfer results for the language considered by Gargov and Goranko, one would have to combine the results presented in this paper with transfer results for the addition of the universal modality (cf. Goranko and Passy [17] for some partial results). In [23] Spaan gives an example of a decidable modal logic from which we can obtain an undecidable logic by enriching its language with the universal modality. In this paper we modify Spaan's example to show that decidability and the finite model property do not transfer if we enrich the modal language with nominals. This resolves Problems 4 and 5 in [12].

For every modal logic L that has a master modality, we define a translation $(\cdot)^*$ from the language of L_H to the language of L and prove that if L admits filtration, then $L \vdash \phi^*$ iff $L_H \vdash \phi$. To use the terminology of Kracht and Wolter [18] and Goguadze, Piazza and Venema [15], we show that L simulates L_H under the interpretation $(\cdot)^*$. As a corollary, we obtain transfer of complexity and (uniform) interpolation from L to L_H, as well as an axiomatization of L_H by adding one simple axiom scheme to the axiomatization of L. We prove similar transfer results for a number of logics without master modality, including the basic modal logic **K**.

Several authors have observed that if a modal logic L admits filtration, then L_H also admits filtration and moreover, if L is in addition finitely axiomatizable, then L_H is decidable (cf in particular [3]). While our simulation argument relies to a large extent on the use of filtration, the results it gives rise to are more fine grained: not only does decidability transfer from L to L_H, but also complexity, (uniform) interpolation and finite axiomatizability. We will also show that even though the logic of symmetric relation **KB** admits filtration, complexity does not transfer from **KB** to its hybrid companion $\mathbf{KB}_H$

The technique developed in this part of the paper allows us to derive short proofs for several results that were proven before by hand, e.g., the complexity of $\mathbf{K}_H$ [1] and the complexity and finite axiomatizability of $\mathbf{PDL}_H$ [21]. Furthermore, we prove that $\mathbf{K}_H$, $\mathbf{S5}_H$, $\mathbf{GL}_H$ and $\mathbf{Grz}_H$ have uniform interpolation over proposition letters. As far as we know, uniform interpolation has not been studied in the context of hybrid logic before. Finally, our results confirm the intuition that adding nominals to a modal logic in many cases does not increase the complexity.

2 Preliminaries

2.1 Hybrid logic in a nutshell

Recall from Part I that the basic hybrid language is the result of extending the modal language with nominals and satisfaction operators. In the present part, we will be concerned with a less expressive hybrid language. The minimal hybrid language is an extension of the basic modal language with nominals. Formally, given a countable set of proposition letters PROP, and a countable set of nominals NOM, and a finite set of modalities MOD, the minimal hybrid language is defined as

$$\phi ::= p \mid i \mid \neg\phi \mid \phi \wedge \psi \mid \langle a \rangle \phi$$

where $p \in$ PROP, $i \in$ NOM and $a \in$ MOD. While the frames that we work with are the same as in ordinary modal logic, we put one extra condition on the models: each nominal must be true at a unique point in the model. In other words, a model is a pair $(\mathfrak{F}, V)$, where $\mathfrak{F}$ is a frame and V is a valuation for $\mathfrak{F}$ with $|V(i)| = 1$ for all $i \in$ NOM. Apart from this extra requirement, no changes are made to the semantics. In particular, the truth definition for nominals is like that of ordinary proposition letters:

$$(\mathfrak{F}, V), w \Vdash i \text{ iff } w \in V(i)$$

The singleton requirement on the valuation of nominals gives rise to new validities. For instance $(i \wedge \Diamond i \wedge \Box p) \to p$ is valid on all frames (if a point is reflexive and all its successors satisfy p, then the point itself satisfies p). Also, using nominals many frame properties are definable that were not definable in the basic modal language. For instance, irreflexivity is defined by $i \to \neg\Diamond i$ (keep in mind that frame validity for hybrid formulas is defined by universal quantification over *hybrid* valuations, i.e., valuations that assign to each nominal a singleton set).

Given a frame class F, we will use $L_H\mathsf{F}$ to denote the set of formulas of the minimal hybrid language that are valid on F (in other words, the *hybrid logic of* F). $L(\mathsf{F})$ is the set of modal formulas valid on F.

Now, suppose we are given a Kripke-complete modal logic L, and let $\mathsf{Fr}L$ be the class of frames on which it is valid (note that L is complete for $\mathsf{Fr}L$). We will use L_H as a shorthand for $L_H\mathsf{Fr}L$, i.e., the hybrid logic of the frame class defined by L. We call L_H the hybrid companion of the modal logic L.

Notice that this is not the only possible way to define hybrid companions for modal logics. In particular, if a modal logic L is complete for several frame classes, the hybrid logics of these classes need not be the same, and one could consider the hybrid logic of other frame classes than the one *defined* by L. Nevertheless, our choice seems a very natural one.

The main question we address in this paper is the following:

Which properties of logics are preserved under passage from L to L_H?

Areces et al. [1] show that the satisfiability problem for the hybrid tense logic $\mathbf{K_t}_H$ is ExpTime-complete. This implies that complexity does not transfer in general, since the satisfiability problem for $\mathbf{K_t}$ is only PSpace-complete. The modal logic of symmetric frames **KB** provides a uni-modal example. The satisfiability problem for **KB** is PSpace-complete [8]. However,

Theorem 2.1 *The satisfiability problem for* $\mathbf{KB}_H$ *is* ExpTime*-complete.*

Proof: For any modal formula ϕ, let $\phi' = i \wedge \Diamond\neg i \wedge \Box\Box\Diamond i \wedge \Box\phi^{\neg i}$, where i is any nominal and $\phi^{\neg i}$ is obtained from ϕ by relativising all modalities with $\neg i$ (i.e., replacing all subformulas of the form $\Diamond\psi$ by $\Diamond(\neg i \wedge \psi)$ and replacing all subformulas of the form $\Box\psi$ by $\Box(\neg i \rightarrow \psi)$). One can easily see that if ϕ' holds at a world w in a symmetric model $\mathfrak{M}$ then ϕ holds globally in the submodel of $\mathfrak{M}$ generated by w, minus the world w itself. It follows that, on symmetric frames, ϕ' is satisfiable iff ϕ is globally satisfiable. The global satisfiability problem for modal formulas on the class of symmetric frames is ExpTime-complete [8]. Hence, the satisfiability problem of $\mathbf{KB}_H$ is ExpTime-hard. That the problem is inside ExpTime follows from the fact that converse **PDL** with nominals is in ExpTime [10]. To see this, with every hybrid formula ϕ we associate a formula $\phi^{\sim}$ in the language of converse **PDL** by replacing every occurrence of $\Diamond$ in ϕ by $\langle(a \cup a^{\sim})^*\rangle$ and replacing every occurrence of $\Box$ in ϕ by $[(a \cup a^{\sim})^*]$, for some fixed atomic program a. Then ϕ is satisfiable on symmetric frames iff $\phi^{\sim}$ is satisfiable, and since converse **PDL** with nominals is in ExpTime, we obtain that $\mathbf{KB}_H$ is also in ExpTime. □

Next, we will show that decidability and the finite model property do not transfer either. Consider the bi-modal language with modalities $\Diamond_1$ and $\Diamond_2$, and let L be the normal modal logic axiomatized by the following Sahlqvist axioms.

$\bigwedge_{1\le k\le 3}\Diamond_1 p_k \rightarrow \bigvee_{1\le k<l\le 3}\Diamond_1(p_k \wedge p_l)$	(at most 2 R_1-successors)
$\bigwedge_{1\le k\le 4}\Diamond_1\Diamond_1 p_k \rightarrow \bigvee_{1\le k<l\le 4}\Diamond_1\Diamond_1(p_k \wedge p_l)$	(at most 3 two-step R_1-successors)
$p \rightarrow \Box_2\Diamond_2 p$	(R_2 is symmetric)

Proposition 2.2 *L has the finite model property and is decidable.*

Proof: First, consider the uni-modal logic axiomatized by the first two axioms. This is a subframe logic (i.e., its class of frames is closed under subframes). It follows by [7, Theorem 11.20] that this logic has the finite model property and is decidable. Now consider the uni-modal logic **KB** given by the last axiom. This logic is complete for the class of symmetric frames, has the finite model property [7] and has a PSpace-complete satisfiability problem [8]. Since decidability and the finite model property are preserved under fusions [11], the result follows. □

Proposition 2.3 *L_H is undecidable and lacks the finite model property.*

Proof: For any uni-modal formula ϕ with modality $\Diamond_1$, let $\phi' = i \wedge \Diamond_2 i \wedge \Box_2\Box_1\Diamond_2 i \wedge \Box_2\phi$. One can easily see that if ϕ' holds at a world w in a model $\mathfrak{M}$ then ϕ holds globally in the submodel of $\mathfrak{M}$ generated by w along the relation R_1. It follows that ϕ' is satisfiable iff ϕ is globally satisfiable. Global satisfiability of uni-modal formulas on the class of frames in which each point has at most two successors and at most three two-step successors is undecidable [23]. It follows that L_H is undecidable.

By Corollary 3.4 of Part I the hybrid formulas in the language $\mathcal{H}(@)$ (and therefore in the language enriched only with nominals) valid on this class are recursively enumerable. It is well known that every logic which has the finite model property and is recursively enumerable is decidable (see e.g., [4, Theorem 6.15]). L_H is recursively enumerable and undecidable, thus, it must lack the finite model property. □

Using the Thomasson simulation, this example can be turned into a uni-modal example of non-transfer of decidability and the finite model property.

In the remainder of this paper, we provide positive results for a class of logics that includes several well-known non-canonical logics, including **PDL**, **GL** and **Grz**. We show that for this class of logics, complexity, finite axiomatizability, interpolation and uniform interpolation transfer.

2.2 Filtrations

In the rest of the paper, we will mainly be concerned with logics that admit filtration. Now we will briefly recall the idea of filtration.

Let $\mathfrak{M}$ be a model based on a frame $\mathfrak{F} = (W, R)$ and let Σ be a set of formulas closed under subformulas. Define an equivalence relation $\sim_\Sigma$ on W such that for every $w, v \in W$ and $\psi \in \Sigma$:

$$w \sim_\Sigma v \quad \text{if} \quad w \models \psi \Leftrightarrow v \models \psi$$

Denote by $[w]$ the $\sim_\Sigma$-equivalence class containing w and let $W/_{\sim_\Sigma}$ be the set of all $\sim_\Sigma$-equivalence classes of W. Define a valuation V_Σ on $W/_{\sim_\Sigma}$ such that $V_\Sigma(p) = \{[w] : w \models p\}$. The model $\mathfrak{M}/_{\sim_\Sigma} = (W/_{\sim_\Sigma}, R_\Sigma, V_\Sigma)$ is called a filtration of $\mathfrak{M}$ through Σ if R_Σ is a binary relation on $W/_{\sim_\Sigma}$ such that for any $\psi \in \Sigma$ and $w \in W$, we have:

$$\mathfrak{M}, w \models \psi \quad \text{iff} \quad \mathfrak{M}/_{\sim_\Sigma}, [w] \models \psi$$

It is easy to see that $|W/_{\sim_\Sigma}| \leq 2^{|\Sigma|}$. Hence $W/_{\sim_\Sigma}$ is finite whenever Σ is finite. The frame $\mathfrak{F}/_{\sim_\Sigma} = (W/_{\sim_\Sigma}, R_\Sigma)$ is called a filtration of $\mathfrak{F}$ through Σ.

Definition 2.4 *We say that a modal logic L admits filtration if for every formula ϕ there exists a finite set of formulas Σ_ϕ containing all subformulas of ϕ such that for every L-frame $\mathfrak{F} = (W, R)$, every point $w \in W$ and every model $\mathfrak{M} = (\mathfrak{F}, V)$ such that $\mathfrak{M}, w \models \phi$, some filtration of $\mathfrak{F}$ over Σ_ϕ is an L-frame.*

We say that a logic L admits polynomial filtration if it admits filtration and the size of Σ_ϕ is polynomial in the length of ϕ. We say that a modal logic admits simple filtration if it admits filtration and for every formula ϕ we have $\Sigma_\phi = Sub(\phi)$.

Note that since the size of $Sub(\phi)$ is linear in the length of ϕ, every logic that admits simple filtration admits polynomial filtration.

Theorem 2.5 *If L admits (polynomial / simple) filtration, then L_H admits (polynomial / simple) filtration.*[4]

Proof: We simply apply the usual filtration, treating nominals as proposition letters. All that needs to be checked is that the filtrated model is a hybrid model, in other words, that every nominal occurring in a given formula is true at exactly one point. Since each nominal is true at some point in the original model, it must also be true at some point in the filtrated model. Now, suppose that a nominal i is true at two points of the filtrated model, say $[w]$ and $[v]$. Then $w \models i$ and $v \models i$ in the original model, and so $w = v$, which implies that $[w] = [v]$. □

[4] Blackburn [3] and Gargov and Goranko [12, §6.3] prove a similar results for hybrid logics with converse modalities and with the universal modality.

3 The translation

In this section, we present our main preservation results. We define two translations from the minimal hybrid language to the basic modal language and prove that they preserve satisfiability. The first translation applies to logics with a master modality, and the second one applies to a number of logics without a master modality.

3.1 Logics with a master modality

Definition 3.1 *A modal logic L has a master modality [4, p. 371] if there is a modal formula $\phi(p)$ containing only the proposition letter p such that for all models $\mathfrak{M}$ based on an L-frame and all worlds w, $\mathfrak{M}, w \models \phi(p)$ iff p is true somewhere in the submodel of $\mathfrak{M}$ generated by w (equivalently, if p is true at a point reachable from w in a finite number of steps). If a logic has a master modality, we will refer to it as $\Diamond^+$ (more precisely, we will use $\Diamond^+\psi$ as a shorthand for $\phi(\psi)$).*

Fact 3.2

1. *Every logic of bounded depth has a master modality.*
2. *Every extension of* **K4** *has a master modality.*
3. *Every extension of* **K5** *has a master modality.*
4. **PDL** *has a master modality.*[5]
5. *Every extension of* $\mathbf{K4} \times \mathbf{K4}$ *has a master modality.*
6. *Every extension of the tense logic* $\mathbf{K4}_t$ *with trichotomy has a master modality (where trichotomy is the axiom $Pp \wedge Pq \rightarrow P(p \wedge Pq) \vee P(q \wedge Pp) \vee P(p \wedge q)$).*

Proof:

1. Take $\phi = \bigvee_{0 \leq k \leq n} \Diamond^k p$, where n is the bound on the depth.
2. Take $\phi = p \vee \Diamond p$.
3. Take $\phi = p \vee \Diamond p \vee \Diamond\Diamond p$
4. Take $\phi = \langle (\cup_i a_i)^* \rangle p$.
5. Take $\phi = p \vee \Diamond_1 p \vee \Diamond_2 p \vee \Diamond_1 \Diamond_2 p$.
6. Take $\phi = p \vee Pp \vee PFp$

□

Note that **K** does not have a master modality (this can easily be shown by the fact that every modal formula has a finite modal depth, and hence can only talk about a small part of the generated submodel). Similarly, the basic tense logic $\mathbf{K}_t$, the tense logic of transitive frames $\mathbf{K4}_t$ and the logic **KB** of symmetric frames do not have a master modality.

The class of logics we will be working with is the class of logics that have a master modality and that admit filtration. Let L be a Kripke-complete modal logic that has a master modality and that admits filtration. Now we define a translation from the language of L_H to the language of L. That is, we translate every hybrid formula into a modal formula. For a hybrid formula $\phi(i_1, \ldots, i_n)$, let $\phi[\vec{i}/\vec{p_i}]$ denote the formula obtained from ϕ by uniformly replacing

[5] For convenience, we assume that the language contains only finitely many atomic programs. The results of this paper can easily be generalized to **PDL** with infinitely many atomic programs.

each nominal i_k by a distinct new proposition letter p_{i_k}. For a hybrid formula $\phi(i_1, \ldots, i_n)$, let

$$\phi^* = \phi[\vec{i}/\vec{p_i}] \;\wedge \bigwedge_{\substack{1 \leq k \leq n \\ \psi \in \Sigma_{\phi[\vec{i}/\vec{p_i}]}}} \Big(\Diamond^+(p_{i_k} \wedge \psi) \rightarrow \Box^+(p_{i_k} \rightarrow \psi) \Big)$$

where $\Sigma_{\phi[\vec{i}/\vec{p_i}]}$ is the filtration set of $\phi[\vec{i}/\vec{p_i}]$.

Theorem 3.3 *Let L be a logic that has a master modality and that admits filtration. Let ϕ be any hybrid formula. Then ϕ is L_H-satisfiable iff ϕ^* is L-satisfiable.*

Proof:

[⇒] Suppose $(\mathfrak{F}, V), w \models \phi$ with $\mathfrak{F} \in \mathsf{Fr}L$. Let V' be any valuation that agrees with V on all proposition letters occurring in ϕ, and such that $V'(p_{i_k}) = V(i_k)$ for each nominal i_k. Clearly, $(\mathfrak{F}, V'), w \models \phi[\vec{i}/\vec{p_i}]$. The truth of the second conjunct of ϕ^* at w under V' follows directly from the fact that $V'(p_{i_k})$ is a singleton set for each $k = 1, \ldots, n$.

[⇐] Suppose $(\mathfrak{F}, V), w \models \phi^*$ with $\mathfrak{F} = (W, R) \in \mathsf{Fr}L$. Without loss of generality, we can assume that $\mathfrak{F}$ is generated by w (note that ϕ^* is a purely modal formula). Our task is to construct a hybrid model satisfying ϕ.

First, we will filtrate $(\mathfrak{F}, V)$. Let $\Sigma = \Sigma_{\phi[\vec{i}/\vec{p_i}]}$. Since L admits filtration, there exists a model $\mathfrak{M} = (W/_{\sim_\Sigma}, R_\Sigma, V_\Sigma)$ such that $(W/_{\sim_\Sigma}, R_\Sigma) \in \mathsf{Fr}L$ and such that for all $v \in W$ and $\psi \in \Sigma$, $\mathfrak{M}, [v] \models \psi$ iff $(\mathfrak{F}, V), v \models \psi$. In particular, $\mathfrak{M}, [w] \models \phi[\vec{i}/\vec{p_i}]$.

Claim 1 *$V_\Sigma(p_{i_k})$ contains at most one point (for $k = 1, \ldots, n$).*

Proof of claim: Suppose $[v], [v'] \in V_\Sigma(p_{i_k})$. Then $v, v' \in V(p_{i_k})$, by the definition of V_Σ. Since $(\mathfrak{F}, V), w \models \Diamond^+(p_{i_k} \wedge \psi) \rightarrow \Box^+(p_{i_k} \rightarrow \psi)$ for all $\psi \in \Sigma$, it follows that v, v' agree on formulas in Σ. Indeed, if $v \models \psi$ then $w \models \Diamond^+(p_{i_k} \wedge \psi)$, so $w \models \Box^+(p_{i_k} \rightarrow \psi)$ and therefore $v' \models \psi$. Thus, $v \sim_\Sigma v'$ and so $[v] = [v']$. ⊣

If every p_{i_k} is true at *exactly* one point, then the proof is finished, since we can consider $(W/_{\sim_\Sigma}, R_\Sigma)$ to be a hybrid model for ϕ. In general, however, this need not be the case: p_{i_k} could be true nowhere. So, we need to ensure that for every p_{i_k} there is indeed a point where p_{i_k} is true. Let $\mathfrak{G}$ be the disjoint union of two isomorphic copies of $(W/_{\sim_\Sigma}, R_\Sigma)$. For convenience, we will use $[v]_1$ and $[v]_2$ to refer to the two distinct copies of a world $[v] \in W/_{\sim_\Sigma}$. Since $\mathsf{Fr}L$ is modally definable, it is closed under disjoint unions and hence, $\mathfrak{G} \in \mathsf{Fr}L$. Define the valuation V' for $(W/_{\sim_\Sigma}, R_\Sigma)$ so that $V'(p) = \{v_1 \mid v \in V_\Sigma(p)\}$ for each proposition letter p occurring in ϕ, and for each nominal $k = 1, \ldots, n$,

$$V'(p_{i_k}) = \begin{cases} \{[v]_1\} & \text{if } V_\Sigma(p_{i_k}) = \{[v]\} \\ \{[w]_2\} & \text{if } V_\Sigma(p_{i_k}) = \emptyset \end{cases}$$

Intuitively speaking, the only role of the second disjoint copy of $(W/_{\sim_\Sigma}, R_\Sigma)$ is to provide enough points so that we can make each p_{i_k} true somewhere, without affecting the truth of ϕ at $[w]$. Indeed, a simple bisimulation argument shows that $(\mathfrak{G}, V'), [w] \models \phi[\vec{i}/\vec{p_i}]$.

By construction, V' assigns to each p_{i_k} a singleton set. Replacing each p_{i_k} by the corresponding i_k, we therefore obtain a hybrid model again, which furthermore satisfies ϕ at $[w]_1$. We conclude that ϕ is satisfiable on $\mathsf{Fr}L$.

$\square$

Corollary 3.4 *Let L be a complete modal logic that has a master modality and that admits filtration. Let ϕ be any hybrid formula with nominals $i_1, \ldots, i_n$. Then ϕ is L_H-valid iff the formula:*

$$\Big(\bigwedge_{\substack{1 \leq k \leq n \\ \psi \in \Sigma_{\neg\phi}}} \Diamond^+(p_{i_k} \wedge \psi) \rightarrow \Box^+(p_{i_k} \rightarrow \psi) \Big) \rightarrow \phi[\vec{i}/\vec{p_i}]$$

is L-valid

Proof: Suppose $L_H \vdash \phi$, for some formula ϕ with nominals $i_1, \ldots, i_n$. Then $\neg\phi$ is not L_H-satisfiable. Hence, by Theorem 3.3,

$$\Big(\bigwedge_{\substack{1 \leq k \leq n \\ \psi \in \Sigma_{\neg\phi}}} \Diamond^+(p_{i_k} \wedge \psi) \rightarrow \Box^+(p_{i_k} \rightarrow \psi) \Big) \wedge \neg\phi[\vec{i}/\vec{p_i}]$$

is not L-satisfiable, which means that

$$L \vdash \Big(\bigwedge_{\substack{1 \leq k \leq n \\ \psi \in \Sigma_{\neg\phi}}} \Diamond^+(p_{i_k} \wedge \psi) \rightarrow \Box^+(p_{i_k} \rightarrow \psi) \Big) \rightarrow \phi[\vec{i}/\vec{p_i}]$$

$\square$

Remark 3.5 We remark here that many well-known logics are known to have a master modality and to admit polynomial filtration. We list some of them with references for the proofs: **K4**, **K45**, **KD45**, **S4**, **S5**, **K4.2**, **K4.3**, **S4.2**, **S4.3**, **K5**, **K4.1**, **S4.1** [7, §5.3]; **GL** and **PDL** [4, §4.8]; **S5** × **S5** [11]; **Grz** [6]. Moreover, all of these logics except **K5**, **K4.1**, **S4.1**, **PDL** and **Grz** admit simple filtration.

3.2 Logics with shallow axioms

Now we show that even though the basic multi-modal logic $\mathbf{K_n}$ does not have a master modality, $\mathbf{K_n}_H$ admits a satisfiability-preserving translation into $\mathbf{K_n}$. We call a modal formula *shallow* if every occurrence of a proposition letter is in the scope of at most one modal operator. We will show that the preservation result holds for extensions of $\mathbf{K}_n$ with shallow axioms. Note that every non-iterative axiom [19] is shallow, as well as every closed formula (i.e., a formula in which no proposition letters occur).

We will use $\blacklozenge\psi$ as a shorthand for $\bigvee_{\Diamond \in \text{MOD}} \Diamond\psi$, $\blacklozenge^k\psi$ as a shorthand for $\underbrace{\blacklozenge \ldots \blacklozenge}_{k-times}$, and $\blacklozenge^{\leq n}\psi$ as a shorthand for $\bigvee_{0 \leq k \leq n} \blacklozenge^k\psi$.

For a hybrid formula $\phi(i_1, \ldots, i_n)$ let

$$\phi^* \quad = \quad \phi[\vec{i}/\vec{p_i}] \quad \wedge \bigwedge_{\substack{1 \leq k \leq n \\ \psi \in Sub(\phi[\vec{i}/\vec{p_i}])}} \Big(\blacklozenge^{\leq md(\phi)}(p_{i_k} \wedge \psi) \rightarrow \blacksquare^{\leq md(\phi)}(p_{i_k} \rightarrow \psi) \Big)$$

where $md(\phi)$ is the modal depth of ϕ [4, Definition 2.28]. Note that the length of ϕ^* is in general exponential in the length of ϕ, but that in case of uni-modal languages, it is polynomial.

Theorem 3.6 *A hybrid formula ϕ is satisfiable iff its modal translation ϕ^* is satisfiable.*

Proof: The left to right implication is easy to prove. Now suppose that ϕ^* is satisfiable. Let $\mathfrak{M}, w \models \phi^*$, with $\mathfrak{M} = (\mathfrak{F}, V)$ and $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond \in \text{MOD}})$. Without loss of generality, we can assume that $\mathfrak{F}$ is generated by w. Let $R_\blacklozenge = \bigcup_{\Diamond \in \text{MOD}} R_\Diamond$. For every point $v \in W$, let $d_\mathfrak{F}(v)$ be the minimal number of $R_\blacklozenge$-steps in which v is reachable from the root w. Consider the equivalence relation $\sim_{Sub(\phi[\vec{i}/\vec{p_i}])}$. Two worlds stand in this equivalence relation iff they satisfy the same subformulas of $\phi[\vec{i}/\vec{p_i}]$. For any $\sim_{Sub(\phi[\vec{i}/\vec{p_i}])}$-equivalence class $[v]$, choose a representative $f[v] \in [v]$ such that for any $v' \in [v]$ we have $d_\mathfrak{F}(f[v]) \leq d_\mathfrak{F}(v')$. Note that while $f[w] = w$, these representatives are in general not unique. Also note that for every $v \in W$ and $\psi \in Sub(\phi[\vec{i}/\vec{p_i}])$, $\mathfrak{M}, v \models \psi$ iff $\mathfrak{M}, f[v] \models \psi$.

Let $W' = \{f[v] \mid v \in W\}$. For each $\Diamond \in \text{MOD}$, define the relation $R'_\Diamond$ on W' so that $f[u] R'_\Diamond f[v]$ iff there is a $v' \in [v]$ with $f[u] R_\Diamond v'$. Define a valuation V' on W' by letting $f[w] \in V'(p)$ iff $w \in V(p)$ for all $p \in Sub(\phi[\vec{i}/\vec{p_i}])$. Let $\mathfrak{F}' = (W', (R'_\Diamond)_{\Diamond \in \text{MOD}})$ and $\mathfrak{M}' = (\mathfrak{F}', V')$.

Claim 1 *For any $\psi \in Sub(\phi[\vec{i}/\vec{p_i}])$ and any point $v \in W$, $\mathfrak{M}, f[v] \models \psi$ iff $\mathfrak{M}', f[v] \models \psi$*

Proof of claim: By induction on the complexity of ψ. If ψ is a propositional letter, then the claim holds by the definition of V'. The Boolean cases are obvious. Finally, let $\psi = \Diamond\chi$, for some $\Diamond \in \text{MOD}$.

[$\Rightarrow$] Suppose that $\mathfrak{M}, f[v] \models \Diamond\chi$. Then there is a point $u \in W$ such that $f[v] R_\Diamond u$ and $\mathfrak{M}, u \models \chi$. Since $\chi \in Sub(\phi[\vec{i}/\vec{p_i}])$ and $u \sim_{Sub(\phi[\vec{i}/\vec{p_i}])} f[u]$, we have that $\mathfrak{M}, f[u] \models \chi$. By the induction hypothesis, it follows that $\mathfrak{M}', f[u] \models \chi$. Finally, we have that $f[v] R'_\Diamond f[u]$, by the definition of $R'_\Diamond$. Hence, $\mathfrak{M}', f[v] \models \Diamond\chi$.

[$\Leftarrow$] Suppose that $\mathfrak{M}', f[v] \models \Diamond\chi$. Then there is an $f[u] \in W'$ such that $f[v] R'_\Diamond f[u]$ and $\mathfrak{M}', f[u] \models \chi$. By the induction hypothesis, $\mathfrak{M}, f[u] \models \chi$. Also, by the definition of $R'_\Diamond$, there must be a $u' \in [u]$ such that $f[v] R_\Diamond u'$. Since $\chi \in Sub(\phi[\vec{i}/\vec{p_i}])$ and $u' \sim_{Sub(\phi[\vec{i}/\vec{p_i}])} f[u]$, it follows that $\mathfrak{M}, u' \models \chi$. We conclude that $\mathfrak{M}, f[v] \models \Diamond\chi$.

$\dashv$

Let us define $d_{\mathfrak{F}'}$ similar to $d_\mathfrak{F}$. Note that $\mathfrak{F}'$ need not be point-generated anymore. For those $f[v] \in W'$ that are not reachable from the root $f[w] = w$, we let $d_{\mathfrak{F}'}(f[v]) = \infty$.

Claim 2 *$d_\mathfrak{F}(f[v]) \leq d_{\mathfrak{F}'}(f[v])$, for all $v \in W$*

Proof of claim: If $d_{\mathfrak{F}'}(f[v]) = \infty$, the claim obviously holds. Otherwise, the proof proceeds by induction on $d_{\mathfrak{F}'}(f[v])$. The base case, with $d_{\mathfrak{F}'}(f[v]) = 0$, only applies if $f[v] = w$, in which case the claim clearly holds. Next, suppose $d_{\mathfrak{F}'}(f[v]) = n+1$. By definition, there must be a path of the form

$$f[w] = w \xrightarrow{R'_{\Diamond_1}} \cdots \xrightarrow{R'_{\Diamond_n}} f[u] \xrightarrow{R'_{\Diamond_{n+1}}} f[v]$$

for some $\Diamond_1, \ldots, \Diamond_{n+1} \in \text{MOD}$. It follows that $d_{\mathfrak{F}'}(f[u]) \leq n$, and hence by the induction hypothesis, $d_\mathfrak{F}(f[u]) \leq d_{\mathfrak{F}'}(f[u]) \leq n$. Since $f[u] R_{\Diamond_{n+1}}{}' f[v]$, by the definition of $R_\Diamond{}'$ we have that there is a $v' \in [v]$ such that $f[u] R_{\Diamond_{n+1}} v'$. This implies that $d_\mathfrak{F}(v') \leq n+1$. By the definition of f, we know that $d_\mathfrak{F}(f[v]) \leq d_\mathfrak{F}(v')$, because $v' \in [v]$. Therefore, $d_\mathfrak{F}(f[v]) \leq n+1$.
$\dashv$

Claim 3 *For all $k = 1 \ldots n$, there is at most one world $f[v] \in W'$ such that $d_{\mathfrak{F}'}(f[v]) \leq md(\phi)$ and $\mathfrak{M}', f[v] \models p_{i_k}$.*

Proof of claim: Suppose $\mathfrak{M}', f[v] \models p_{i_k}$ and $\mathfrak{M}', f[u] \models p_{i_k}$, with $d_{\mathfrak{F}'}(f[v]), d_{\mathfrak{F}'}(f[u]) \leq md(\phi)$. By Claim 2, $d_{\mathfrak{F}}(f[v]), d_{\mathfrak{F}}(f[u]) \leq md(\phi)$. Furthermore, $\mathfrak{M}, f[v] \models p_{i_k}$ and $\mathfrak{M}, f[u] \models p_{i_k}$. By our initial assumption, $\mathfrak{M}, w \models \phi^*$, hence $f[v] \sim_{Sub(\phi)} f[u]$, which implies that $f[v] = f[u]$. ⊣

From Claim 1, we immediately deduce that $\mathfrak{M}', w \models \phi[\vec{i}/\vec{p_i}]$. The valuation of $p_{i_1}, \ldots, p_{i_n}$ can be restricted to the worlds with depth $\leq md(\phi)$ without affecting the truth of $\phi[\vec{i}/\vec{p_i}]$ at w. In this way, we ensure that every p_{i_k} is true at at most one world. Finally, applying the same argument as in the proof of Theorem 3.3, we conclude that the original hybrid formula ϕ is satisfiable. □

Let L be a modal logic defined by finitely many shallow axioms, and for a hybrid formula $\phi(i_1, \ldots, i_n)$, let

$$\phi^* \quad = \quad \phi[\vec{i}/\vec{p_i}] \ \wedge \bigwedge_{\substack{1 \leq k \leq n \\ \psi \in \Sigma}} \Big(\blacklozenge^{\leq md(\phi)}(p_{i_k} \wedge \psi) \rightarrow \blacksquare^{\leq md(\phi)}(p_{i_k} \rightarrow \psi) \Big)$$

where Σ consists of all subformulas of ϕ plus all closed subformulas of the (finitely many) shallow axioms of L (recall that a modal formula is closed if it contains no proposition letters).

Theorem 3.7 *Let L be any complete modal logic axiomatized by finitely many shallow axioms. A hybrid formula ϕ is L_H-satisfiable iff ϕ^* is L-satisfiable.*

Proof: We use the same construction as in the proof of Theorem 3.6, but now we use a richer filtration set, that also includes all closed subformulas of the shallow axioms of L. It suffices to show that the constructed frame $\mathfrak{F}'$ is an L-frame. Let V' be a valuation on $\mathfrak{F}'$, and let $x \in W'$ be such that $(\mathfrak{F}', V'), x \models \phi$. Define V on $\mathfrak{F}$ such that $v \in V(p)$ iff $f[v] \in V'(p)$. We claim that for all shallow axioms χ of L and for all $v \in W$, $(\mathfrak{F}, V), f[v] \models \chi$ iff $(\mathfrak{F}', V'), f[v] \models \chi$.

This, we prove by induction on χ. Note that χ is shallow, and hence we may assume that χ is generated by the following recursive definition:

> $\chi ::= \top \mid p \mid \neg\chi \mid \chi_1 \wedge \chi_2 \mid \Diamond\psi$, where ψ is any Boolean combination of proposition letters and closed formulas.

The only non-trivial case in the induction is when χ is of the form $\Diamond\psi$ where ψ is a Boolean combination of proposition letters and closed formulas. In this case, we reason as follows.

[⇒] Suppose $(\mathfrak{F}, V), f[v] \models \Diamond\psi$. Then there is a $u \in W$ such that $f[v] R_\Diamond u$ and $(\mathfrak{F}, V), u \models \psi$. By the definition of V and the fact that all closed subformulas of ψ are in the filtration set, it follows that $(\mathfrak{F}', V'), f[u] \models \psi$. By definition of $R'_\Diamond$, $f[v] R'_\Diamond f[u]$. Hence, $(\mathfrak{F}', V'), f[v] \models \Diamond\psi$.

[⇐] Suppose $(\mathfrak{F}', V'), f[v] \models \Diamond\psi$. Then there is an $f[u] \in W'$ such that $(\mathfrak{F}', V'), f[u] \models \psi$ and $f[v] R'_\Diamond f[u]$. By definition of $R'_\Diamond$, there is a $u' \in [u]$ such that $f[v] R_\Diamond u'$. By the definition of V and the fact that all closed subformulas of ψ are in the filtration set, it follows that $(\mathfrak{F}, V), u' \models \psi$. Hence, $(\mathfrak{F}, V), f[v] \models \Diamond\psi$.

□

This covers logics axiomatized using reflexivity ($\Box p \rightarrow p$), totality ($\Diamond\top$) and bounded width ($\mathrm{bw}_n \equiv \bigwedge_{1 \leq k \leq n} \Diamond p_k \rightarrow \bigvee_{1 \leq k < l \leq n} \Diamond(p_k \wedge p_l)$).

Note again that the length of ϕ^* is in general exponential in the length of ϕ, but that in case of uni-modal languages, it is polynomial.

Corollary 3.8 *Let L be a complete uni-modal logic axiomatizable by finitely many shallow axioms. Let ϕ be any hybrid formula with nominals $i_1, \dots i_n$. Then ϕ is L_H-valid iff the following formula is L-valid:*

$$\Big(\bigwedge_{\substack{1\leq k\leq n \\ \psi\in\Sigma}} \blacklozenge^{\leq md(\phi)}(p_{i_k} \wedge \psi) \to \blacksquare^{\leq md(\phi)}(p_{i_k} \to \psi) \Big) \to \phi[\vec{i}/\vec{p_i}]$$

where Σ consists of all subformulas of ϕ plus all closed subformulas of the (finitely many) shallow axioms of L

Proof: As for Corollary 3.4. □

4 Applications of the translation

For the sake of brevity, in this section all results are stated without proof. The reader is referred to [2] for the proofs.

From Theorem 3.3 and 3.7, together with the observation that for logics admitting polynomial filtration, the length of ϕ^* is polynomial in the length of ϕ, we obtain the following transfer result for complexity.

Corollary 4.1 *Let L be a complete modal logic satisfying one of the following conditions:*

(a) L has a master modality and admits polynomial filtration.

(b) L is a uni-modal logic defined by finitely many shallow axioms.

Then L_H-satisfiability is polynomially reducible to L-satisfiability.

Hence, if a modal logic L satisfies condition (a) or (b), and if C is any complexity class closed under polynomial reductions, such as NP, PSpace, ExpTime, NExpTime, 2ExpTime, etc., then L-satisfiability is in (complete for) C iff L_H is in (complete for) C.

Note that Corollary 4.1 cannot be easily generalized, since **KB**, the uni-modal logic of symmetric frames, admits polynomial filtration, yet by Theorem 2.1, $\mathbf{KB}_H$ is ExpTime-complete, whereas **KB** is only PSpace-complete.

Next, we will discuss the issue of transfer of interpolation and uniform interpolation. For any hybrid formula ϕ, let $\mathbb{P}(\phi)$ and $\mathbb{N}(\phi)$ denote the set of proposition letters and nominals, respectively, occurring in ϕ.

Definition 4.2 (Interpolation for hybrid logics) *A hybrid logic L has interpolation over proposition letters if for all formulas ϕ and ψ such that $L \vdash \phi \to \psi$, there is a formula θ such that $L \vdash \phi \to \theta$, $L \vdash \theta \to \psi$ and $\mathbb{P}(\theta) \subseteq \mathbb{P}(\phi) \cap \mathbb{P}(\psi)$.*

Note that according to this definition, θ might contain nominals occurring in ϕ but not in ψ or vice versa. It seems more natural to require that the nominals occurring in the interpolant θ should occur both in ϕ and in ψ. However, it turns out that almost all hybrid logics lack this strong form of interpolation [24].

Recall that a modal logic *admits simple filtration* if it admits filtration and for every formula ϕ we have $\Sigma_\phi = Sub(\phi)$. For logics admitting simple filtration, interpolation transfers.

Theorem 4.3 *Let L be a complete modal logic satisfying one of the following conditions:*

(a) L has a master modality and admits simple filtration.

(b) L is defined by finitely many shallow axioms.

If L has interpolation, then L_H has interpolation over proposition letters.

Definition 4.4 (Uniform interpolation for hybrid logics) *A hybrid logic L has uniform interpolation over proposition letters if for each formula ϕ and each finite set of proposition letters $P \subseteq \mathbb{P}(\phi)$, there is a formula ϕ_P such that*

- *$\mathbb{P}(\phi_P) \subseteq P$, and*
- *For all formulas ψ, if $\mathbb{P}(\psi) \cap \mathbb{P}(\phi) \subseteq P$ and $\mathbb{N}(\psi) \subseteq \mathbb{N}(\phi)$, then $L \vdash \phi \to \psi$ iff $L \vdash \phi_P \to \psi$.*

In contrast to what one might expect, the uniform interpolant ϕ_P does not apply to formulas ψ that contain nominals not occurring in ϕ.

Theorem 4.5 *Let L be a complete modal logic satisfying one of the following conditions:*

(a) L has a master modality and admits simple filtration.
(b) L is defined by finitely many shallow formulas.

If L has uniform interpolation, then L_H has uniform interpolation over proposition letters.

It is known that **K**, **GL**, **S5** and **Grz** have uniform interpolation (see [27] and [14]). From Theorem 4.5 and the fact that **GL** and **S5** admit simple filtration, it follows immediately that $\mathbf{K}_H$, $\mathbf{S5}_H$ and $\mathbf{GL}_H$ have uniform interpolation over proposition letters. **Grz** does not admit simple filtration. Nevertheless, the same technique can be applied to $\mathbf{Grz}_H$ as well.

Theorem 4.6 $\mathbf{Grz}_H$ *has uniform interpolation over proposition letters.*

Definition 4.7 *For any modal logic L, let L^H be the closure of*

$$L \cup \{\Diamond^n(i \wedge \phi) \to \Box^m(i \to \phi) \mid i \text{ is a nominal, } \phi \text{ is a formula and } n, m \in \omega\}$$

under Modus Ponens, Necessitation and Uniform Substitution for proposition letters.[6]

For logics that have a master modality, the set of axioms from Definition 4.7 collapses to the single axiom scheme $\Diamond^+(i \wedge \phi) \to \Box^+(i \to \phi)$.

Theorem 4.8 *Let L be a complete modal logic satisfying one of the following conditions:*

(a) L has a master modality and admits filtration.
(b) L is axiomatized by finitely many shallow modal formulas.

Then $L_H = L^H$.

Remark 4.9 Suppose L is a logic that satisfies our conditions (has a master modality and admits filtrations). Furthermore, suppose L is complete for a frame class F. In general we cannot conclude from our results that L^H is complete for F. All we know is that L^H is complete for $\mathsf{Fr}(L)$. Consider the case of **GL**. As is well known, **GL** is not only complete for the class of transitive conversely well-founded frames (which it defines), but also for the class of finite transitive irreflexive trees (*finite trees* for short). By Theorem 4.8 we know that $\mathbf{GL}^H$ is complete for the class of transitive conversely well-founded frames. As it turns out, though, $\mathbf{GL}^H$ is *not* complete for the class of finite trees: the formula

$$\Diamond(p \wedge \Diamond i) \ \wedge \ \Diamond(q \wedge \Diamond i) \ \to \ \Diamond(p \wedge \Diamond q) \ \vee \ \Diamond(q \wedge \Diamond p) \ \vee \ \Diamond(p \wedge q)$$

is valid on finite trees but is not valid on the class of transitive conversely well-founded frames. Hence, it is not derivable in $\mathbf{GL}^H$. We conjecture that if this formula is added as an axiom to $\mathbf{GL}^H$, the resulting logic is complete for finite trees.

[6] By *Uniform Substitution for proposition letters*, we mean that proposition letters (not nominals) can be replaced uniformly by arbitrary formulas [4].

5 Conclusions

In this part of the paper, we addressed the question of which properties transfer under passage from a modal logic to its hybrid companion. Using the technique of Spaan [23], we showed that complexity, decidability and the finite model property do not transfer in general, thus resolving Problems 4 and 5 in [12]. Next, we provided a translation from hybrid logic to modal logic. We showed that this translation preserves validity for logics that have a master modality and that admit filtration. Using this translation, we derived transfer results for complexity, interpolation and axiomatic completeness. We also gave a translation that applies to logics axiomatized by shallow formulas, i.e., formulas in which every occurrence of a proposition letter is under the scope of at most one modal operator.

As far as we are aware, Corollary 4.1 is the first general complexity result for hybrid logics. Uniform interpolation has also not been studied before in the context of hybrid logic. Finally, while several non-elementary hybrid logics have been investigated before (in particular $\mathbf{PDL}_H$ [21, 13, 10]), all general completeness results we are aware of [12, 4, 26] only apply to elementary, or at least canonical, logics.

There are still many questions remaining. The study of this topic could be developed in three directions: (1) find other classes (or extend the class of logics we are working with) for which the translation works, (2) see which other properties transfer from L to L_H, (3) generalize these results to richer hybrid languages. Regarding the first point, our result concerning the logic of symmetric frames **KB** suggests that the class of logics to which the translation applies cannot be easily generalized. Concerning the second point, it was proved in Gargov and Goranko [12] and in Part I of this paper that canonicity transfers. Interesting questions are whether interpolation, compactness and the Beth property transfer in general. With respect to the third point, the authors are currently working on generalizations of the present results to hybrid languages with @-operators.

Finally, note that we could have defined the hybrid companion of a modal logic L to be L^H rather than L_H. In that case, we clearly get axiomatizations for free, and this definition would apply also to incomplete modal logics. On the other hand, transfer of completeness is in this case not straightforward anymore, and is indeed an interesting question.

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