# Guarded fragments with constants

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**Abstract.** We prove ExpTime-membership of the satisfiability problem for loosely ∀-guarded first-order formulas with a bounded number of variables and an unbounded number of constants. Guarded fragments with constants are interesting by themselves and because of their connection to hybrid logic.

**Keywords:** Guarded fragment, Hybrid Logic, Complexity

## 1. Introduction

The guarded fragment of first-order logic was first introduced by Andréka, Van Benthem and Németi [1], who proved that it is decidable and that it has a number of other desirable properties. Van Benthem [9] improved on this by generalizing the guarded fragment to the loosely guarded fragment and showing that the latter is still decidable.

Grädel [4] further improved on these results in a number of ways. He generalized the guarded and loosely guarded fragments by allowing constants and the equality symbol to occur in formulas (but no function symbols of positive arity), and subsequently proved the following:

THEOREM 1. (Grädel [4]). *The satisfiability problem for loosely guarded formulas is 2*ExpTime*-complete. The same problem is* ExpTime*-complete for loosely guarded relational formulas with a bounded number of variables, and for guarded relational formulas with a bound on the arity of the relation symbols.*

With a relational formula, we mean a formula that contains no constants (function symbols of positive arity were already excluded).

Grädel [4] indicates that his results also apply to loosely ∀-guarded formulas (i.e., formulas of which only the universal quantifiers are loosely guarded), although the proof of this is not clearly spelled out.[1]

[1] Marx [6] does explicitly state and prove the *decidability* of the satisfiability problem for loosely ∀-guarded formulas.

In the present paper we improve Grädel's results by showing that the qualification *'relational'* in the above theorem may be dropped. Concretely, we prove the following.

THEOREM 2. *The satisfiability problem for loosely ∀-guarded formulas with a bounded number of variables and for guarded formulas with a bounded arity is* ExpTime*-complete.*

To appreciate the additional value of Theorem 2, we must return to the original motivation behind the guarded fragment. The guarded fragment was invented in order to explain and generalize the nice computational and model theoretic properties of the modal language. The key observation is that modal operators express a guarded form of quantification, where the accessibility relations are the guards. For explaining *decidability* results in modal logic, the first part of Theorem 1 often suffices. However, in order to explain *low complexity*, a more refined analysis is needed. Consider for instance the global consequence problem for modal formulas (*does every model that globally satisfies* $\phi$ *globally satisfy* $\psi$*?*). This is an ExpTime-complete problem. To understand why this problem is in ExpTime, it suffices to observe that global truth of a modal formula $\phi$ can be expressed by means of a guarded first-order formula with only two variables, namely $\forall x.(x = x \rightarrow ST_x(\phi))$.[2] This shows the importance of bounded variable guarded fragments.

Recently there has been much interest in modal languages with *nominals* [2]. Nominals are special proposition letters that denote singleton sets. If we translate modal formulas containing nominals into first order logic, then we arrive in the two-variable guarded fragment with an unlimited number of constants. Theorem 1 will therefore not allow us to prove, say, that the global consequence problem for modal formulas with nominals is in ExpTime. Theorem 2 does, and it thereby broadens the application of guarded fragments to the field of hybrid logic (this is a common name for the family of modal languages with nominals and related machinery).

Interestingly, and as an aside, recent results on hybrid logic have also found applications for guarded fragments with constants. One such application concerns the interpolation property. When the guarded fragment was introduced, it was hoped that it has interpolation [1]. This turned out not to be the case [5]. In [7], it was shown using results from hybrid logic that, in some sense, every extension of the guarded fragment with constants that has interpolation has already full first-order expressive power.

[2] Here, $ST$ refers to the well-known Gabbay-style standard translation of modal formulas to first-order formulas, that uses only two variables.

## 2. Preliminaries

We will consider first-order languages with arbitrarily many relation symbols of any arity, constants and equality, but without function symbols of arity greater than zero. A first-order formula $\phi$ of such a language is called *guarded* if it is built up from atomic formulas using the Boolean connectives and guarded quantifiers of the form $\exists x_1 \ldots x_n.(\pi \wedge \psi)$ or $\forall x_1 \ldots x_n.(\pi \rightarrow \psi)$, where $\pi$ is an atomic formula and the free variables of $\psi$ all occur in $\pi$. A formula is called $\forall$*-guarded* if it is built up from atomic formulas and negated atomic formulas using conjunction, disjunction, ordinary existential quantifiers and guarded universal quantifiers. Note that the guards $\pi$ may be atomic equality statements. In particular, if a guarded formula $\phi$ has only one free variable $x$, then $\exists x.(x = x \wedge \phi)$ and $\forall x.(x = x \rightarrow \phi)$ are guarded formulas. These formulas are equivalent to $\exists x.\phi$ and $\forall x.\phi$, respectively.

The loosely guarded fragment is an extension of the guarded fragment. A first-order formula $\phi$ is called *loosely guarded* if it is built up from atomic formulas using the Boolean connectives and loosely guarded quantifiers of the form $\exists x_1 \ldots x_n.(\pi \wedge \psi)$ or $\forall x_1 \ldots x_n.(\pi \rightarrow \psi)$, where $\pi$ is conjunction of atomic formulas, such that every quantified variable $x_i$ co-occurs with every free variable $y \neq x_i$ of $\psi$ in some conjunct of $\pi$. A formula is called *loosely* $\forall$*-guarded* if it is built up from atomic formulas and negated atomic formulas using conjunction, disjunction, ordinary existential quantifiers and loosely guarded universal quantifiers. Note that if a loosely guarded formula $\phi$ has only one free variable $x$, then $\exists x.(\top \wedge \phi)$ and $\forall x.(\top \rightarrow \phi)$ are loosely guarded.

For any formula $\phi$, $\textsc{width}(\phi)$ will be the maximal number of free variables of a subformula of $\phi$, i.e., $\textsc{width}(\phi)$ is the largest natural number $n$ such that $\phi$ has a subformula with $n$ free variables.

Grädel [4] proved his main decidability and complexity results for guarded formulas using the following normal form.

DEFINITION 1. *A (loosely)* $\forall$*-guarded formula is in* normal form *if it is of the form*

$$\exists \vec{x}.P(\vec{x}) \wedge \bigwedge_{i \in I} \forall \vec{x}.(\pi_i(\vec{x}) \rightarrow \exists \vec{y}.\phi_i(\vec{x}, \vec{y}))$$

*where, for each* $i \in I$*, the variables* $\vec{x}, \vec{y}$ *are distinct,* $\pi_i$ *is a (loose) guard and* $\phi_i(\vec{x}, \vec{y})$ *is a quantifier-free formula.*

Grädel showed that every (loosely) guarded formula can be translated in polynomial time into an equisatisfiable (loosely) $\forall$-guarded formula in normal form. A slight variation of Grädel's proof works for (loosely)

$\forall$-guarded sentences, thus turning it into a true normal form theorem for (loosely) $\forall$-guarded formulas (cf. the full version of this paper for more details [8]):

PROPOSITION 1. ([4]). *Every (loosely) $\forall$-guarded formula $\phi$ can be transformed in polynomial time into an equisatisfiable (loosely) $\forall$-guarded sentence $\chi$ in normal form. Moreover,* $\textsc{width}(\chi) \leq \textsc{width}(\phi)$.

Next, Grädel proved the following complexity result for such sentences.

PROPOSITION 2. ([4]). *The satisfiability problem for loosely $\forall$-guarded sentences in normal form is 2*EXPTIME*-complete. It is* EXPTIME*-complete if there is a bound on the width of the sentence.*

Note that Theorem 1 follows immediately from these results. In fact, it follows that the satisfiability problem for loosely $\forall$-guarded formulas is 2EXPTIME-complete, and it is EXPTIME-complete for loosely $\forall$-guarded relational formulas with a bounded number of variables.

Incidentally, the constraints of bounded width and of bounded number of variables in a first-order formula are equivalent. as in the following folklore result (a proof of which can be found in the full version of this paper [8]).

PROPOSITION 3. *For $k \in \mathbb{N}$, every first-order formula $\phi$ of width $k$ can be transformed in polynomial time into an equivalent formula containing only $k$ variables.*

## 3. Eliminating constants

Most results on guarded formulas have been stated only for relational first-order formulas, i.e., formulas not containing constants. In this section, we will discuss how these results can be applied to formulas containing constants. Let $\textsc{cons}(\phi)$ be the set of constants occurring in $\phi$. Grädel [4] proved the following.[3]

PROPOSITION 4. *Every (loosely) $\forall$-guarded formula $\phi$ can be transformed in polynomial time into an equisatisfiable relational (loosely) $\forall$-guarded formula $\chi$, such that* $\textsc{width}(\chi) \leq \textsc{width}(\phi) + |\textsc{cons}(\phi)|$.

[3] Strictly speaking, Grädel's proof for this proposition is flawed, since his translation does not correctly handle formulas containing equality. However, this problem can easily be fixed.

For complexity reasons, we have a particular interest in formulas with a bounded width. Unfortunately, for such formulas $\phi$, Proposition 4 does not imply a bound on the width of $\chi$. We will present another method to eliminate constants, that allows us to circumvent this problem.

However, first we will prove a technical lemma. The lemma shows that, in the case of loosely guarded formulas, the arity of the relation symbols occurring in the formula may be bounded by the width. For any formula $\phi$, let $\textsc{maxarity}(\phi)$ denote the highest arity of a relation symbol occurring in $\phi$.

LEMMA 1. *Every loosely $\forall$-guarded formula $\phi$ can be transformed in polynomial time into an equisatisfiable loosely $\forall$-guarded formula $\chi$ in normal form, such that* $\textsc{width}(\chi) \leq \max\{\textsc{width}(\phi), 2\}$ *and* $\textsc{maxarity}(\chi) \leq \max\{\textsc{width}(\phi), 2\}$.

*Proof.* The proof proceeds in two steps. First, we reduce the arity of the relation symbols occurring in $\phi$ to two, using a familiar trick: we replace each $n$-ary relation symbol by $n$ distinct binary relation symbols. Then, we write the resulting formula in normal form. The latter step might increase the arity of the relation symbols again, but it will still be bounded by the width of the formula.

Let $\phi$ be any loosely $\forall$-guarded formula. For each $n$-ary relation symbol $R$ occurring in $\phi$, with $n > 2$, introduce $n + 1$ new binary relation symbols, $R_0, \ldots, R_n$. These relation symbols will be used to encode the tuples that stand in the relation $R$: a tuple $\langle d_1, \ldots, d_n \rangle$ will be thought to stand in the relation if each pair $\langle d_\ell, d_m \rangle$ stands in the $R_0$ relation $(1 \leq \ell, m \leq n)$, and there exists an element $e$ such that $\langle e, d_\ell \rangle \in R_\ell$ for $1 \leq \ell \leq n$.

Replace each subformula of $\phi$ of the form $R(t_1, \ldots, t_n)$ that is not inside a guard by

$$\bigwedge_{1 \leq \ell, m \leq n} R_0(t_\ell, t_m) \ \wedge \ \exists u. \bigwedge_{1 \leq \ell \leq n} R_\ell(u, x_\ell)$$

If $\phi$ has a subformula of the form $\forall \vec{x}(\pi \to \psi)$, where the guard $\pi$ contains a conjunct of the form $R(t_1, \ldots, t_n)$, then replace that conjunct by $\bigwedge_{1 \leq \ell, m \leq n} R_0(t_\ell, t_m)$, and replace $\psi$ by $\exists u.(\bigwedge_{1 \leq \ell \leq n} R_\ell(u, x_\ell) \wedge \top) \to \psi$.

The resulting formula contains no relation symbols of arity greater than 2, and it is satisfiable iff the original formula $\phi$ is satisfiable. Furthermore, the width of the resulting formula is at most $\max\{\textsc{width}(\phi), 2\}$.

Finally, we apply Proposition 1 to bring the resulting formula into normal form. Inspection of the proof of Proposition 1 shows that the arity of the relation symbols added during the normal form translation is bounded by the width of the input formula. Hence, we end up with a formula with the desired properties. □

The following proposition is central to this paper.

PROPOSITION 5. *Fix a natural number $k \geq 2$. Every loosely $\forall$-guarded formula $\phi$ of width at most $k$ can be transformed in polynomial time into an equisatisfiable relational loosely $\forall$-guarded formula $\chi$ of width at most $k$.*

*Proof.* Consider any loosely $\forall$-guarded formula $\phi$ of width at most $k$. By Lemma 1, we may assume that $\phi$ is in normal form and that $\textsc{maxarity}(\phi) \leq k$.

Let $\textsc{cons}$ be the set of constants occurring in $\phi$. Consider any $n$-place relation symbol $R$ occurring in $\phi$, except for equality, and consider any partial function $f : \{1, \ldots, n\} \hookrightarrow \textsc{cons}$ assigning constants to the argument positions of $R$. For each such $R$ and $f$, introduce a new relation symbol $R_f$ with arity $n - |dom(f)|$, where $dom(f)$ is the set of all $k \in \{1, \ldots, n\}$ for which $f(k)$ is defined. For example, if $R$ is a ternary relation symbol and $f = \{(1, c), (3, d)\}$, then $R_f$ is a unary relation symbol, and we will also denote it by $R_{c\bullet d}$ (where the "black hole" $\bullet$ indicates that $f(2)$ is undefined). The intended interpretation of $R_{c\bullet d}(x)$ will be the same as $R(c, x, d)$. Also, for each pair of constants $c, d$, introduce a nullary relation symbol $E_{cd}$.

We will now eliminate all constants, with the help of these new relation symbols. For any sequence of variables $\vec{x}$, let $T(\vec{x})$ be the set of all partial functions from $\{\vec{x}\}$ to $\textsc{cons}$ (including the empty function). Note that there are $(|\textsc{cons}| + 1)^{|\vec{x}|}$ such functions. For each $\tau \in T(\vec{x})$ and formula $\psi$, let $\psi^\tau$ be the result of replacing each occurrence of a variable $x \in dom(\tau)$ by $\tau(x)$. Finally, let $\phi^*$ be obtained from $\phi$ by means of the following procedure.

1. Replace each subformula of the form $\forall\vec{x}.\psi$ by $\bigwedge_{\tau \in T(\vec{x})} \forall\vec{x}.\psi^\tau$, and replace each subformula of the form $\exists\vec{y}.\psi$ by $\bigvee_{\tau \in T(\vec{y})} \exists\vec{y}.\psi^\tau$. [4]

2. Replace each atomic formula of the form $R(c_1, \ldots, c_n, x_1, \ldots, x_m)$ by $R_{c_1 \ldots c_n \bullet \ldots \bullet}(x_1, \ldots, x_m)$ (and similarly for other permutations)

3. Replace each atomic formulas of the form $c = d$ by $E_{cd}$, and replace each atomic formula of the form $x = c$ or $c = x$ by $\bot$.

[4] Note that this will only polynomially increase the length of the formula, due to the fact that the width and the quantifier depth of $\phi$ are both bounded (keep in mind that $\phi$ is in normal form).

Let $\chi$ be the conjunction of $\phi^*$ with

$$\bigwedge_{c \in \text{CONS}} E_{cc} \wedge \bigwedge_{c,d \in \text{CONS}} E_{cd} \to E_{dc} \wedge \bigwedge_{c,d,e \in \text{CONS}} E_{cd} \wedge E_{de} \to E_{ce}$$

and all formulas of the form

$$\forall x_1 \ldots x_m . \Big( R_{c_1 \ldots c_\ell \ldots c_n \bullet \ldots \bullet}(x_1, \ldots, x_m) \to \\ (E_{c_\ell d} \to R_{c_1 \ldots d \ldots c_n \bullet \ldots \bullet}(x_1, \ldots, x_m)) \Big)$$

(including all permutations of the sequence $c_1, \ldots, c_n x_1, \ldots, x_m$).[5]

Clearly, $\chi$ does not contain any constants, and is loosely $\forall$-guarded. Furthermore, the length of $\chi$ is polynomial in the length of $\phi$, and $\chi$ can be obtained from $\phi$ in polynomial time.

Finally, we claim that $\chi$ is satisfiable iff $\phi$ is satisfiable. One direction of this claim is easy: a model for $\phi$ is easily turned into a model for $\chi$. As for the other direction, every model $M$ satisfying $\chi$ can be turned into a model $M'$ for $\phi$ in the following way: define an equivalence relation on the set CONS by putting $c \sim d$ iff $M \models E_{cd}$, extend the domain of $M$ with one element for each equivalence class, and extend the relations to the new elements in the obvious way: $([c_1], \ldots, [c_n], e_1, \ldots, e_m) \in R$ iff $(e_1, \ldots, e_m) \in R_{c_1 \ldots c_n \bullet \ldots \bullet}$, and likewise for other permutations. It is easily seen that the resulting model $M'$ satisfies $\phi$. □

Note that the translation used in the above proof is polynomial only provided that the width of the input formula is bounded by a constant. Unlike Grädel's translation, it is in general exponential.

Theorem 2 now follows: the first half follows immediately from Propositions 1, 2 and 5 (trivially, the width of a formula is bounded by the number of variables occurring in it). For the second half, it suffices to observe that the width of a guarded formula is bounded by the arity of the relation symbols occurring in it.

Incidentally, in general, the latter does not hold for $\forall$-guarded formulas or loosely guarded formulas. Indeed, by a similar argument as used in the proof of Lemma 1, the satisfiability problem for loosely guarded formulas with arity at most 2 is already as hard as the satisfiability problem for loosely guarded formulas in general, i.e., 2EXPTIME-complete.

[5] The number of such formulas is in the order of

$$|\text{REL}(\phi)| \cdot \left( |\text{CONS}(\phi)|^{\text{MAXARITY}(\phi)} \right)$$

where REL($\phi$) is the set of relation symbols occurring in $\phi$. This is polynomial in the length of $\phi$, given that MAXARITY($\phi$) is bounded by $k$.

## 4. Discussion

We finish by discussing two open questions. The first question is:

> *What is the complexity of the satisfiability problem for ∀-guarded formulas with bounded arity?*

Note that the answer to this question does not depend on the presence of constants. Our conjecture is that this problem is ExpTime-complete.

A second interesting question would be the following:

> *Classify, in the style of Börger et al. [3], the quantifier patterns $\pi$ for which the satisfiability problem for sentences consisting of a sequence of quantifiers conform $\pi$ followed by a guarded formula, is decidable.*

The satisfiability problem for $\pi = \exists^*\forall\exists^*$ is still decidable, as can be seen by replacing the outermost existentially quantified variables by constants and guarding the universal quantifier by an identity statement of the form $x = x$. On the other hand, $\pi = \forall^3$ is already a conservative reduction class, as follows from results of Grädel [4]. What about $\pi = \exists^*\forall^2\exists^*$?

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