A Tableau System for Quantified Hybrid Logic

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Abstract

We present a (sound and complete) tableau calculus for Quantified Hybrid Logic (*QHL*). *QHL* is an extension of orthodox quantified modal logic: as well as the usual $\Box$ and $\Diamond$ modalities it contains names for (and variables over) states, operators $@_s$ for asserting that a formula holds at a named state, and a binder $\downarrow$ that binds a variable to the current state. The first-order component contains equality and rigid and non rigid designators. As far as we are aware, ours is the first tableau system for *QHL*.

The tableau calculus is highly flexible. We only present it for the constant domain semantics, but slight changes render it complete for varying, expanding or contracting domains. Moreover, completeness with respect to specific frame classes can be obtained simply by adding extra rules or axioms (this can be done for every first-order definable class of frames which is closed under and reflects generated subframes). We briefly discuss such theoretical issues at the end of the paper, but the main aim of the present paper is simply to give an example driven introduction to the tableau system.

1 Introduction

Hybrid logic is an extension of modal logic in which it is possible to name states and to assert that a formula is true at a named state. Hybrid logic uses three fundamental tools to do this: nominals, satisfaction operators, and the $\downarrow$-binder. Nominals are special propositional symbols that are true at precisely one state in any model: nominals 'name' the unique state they are true at. A satisfaction operator has the form $@_s$ where s is a nominal. A formula of the form $@_s\phi$ asserts that ϕ is true at the state named by the nominal s. Finally, a formula of the form $\downarrow s.\phi$ binds all occurrences of the nominal s in ϕ to the current state of evaluation — that is, it makes s a name for the current state. (Actually, so that we don't have to worry about accidental binding in the course of tableau proofs, we shall distinguish between ordinary nominals, which cannot be bound, and 'state variables' which are essentially bindable nominals.)

Hybrid logic has a lengthy history (see the webpage `www.hylo.net` for further information), and over the years it has become clear that adding the hybrid apparatus of nominals (and state variables), satisfaction operators, and $\downarrow$ to modal logic often results in systems

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with better logical properties than the original. But most previous work on hybrid logic has examined the effects of hybridizing *propositional* modal logics. What about *quantified* (first-order) hybrid logic?

In fact, strong evidence already exists that quantified hybrid logic (*QHL*) is also better behaved logically than orthodox quantified modal logic. In [2], the only recent paper devoted to the topic, it is shown that a very general interpolation theorem holds in *QHL* (as is well known interpolation almost never holds in orthodox quantified modal logic [4]). The purpose of the present paper is to show that *QHL* is well behaved in another respect: just as in the propositional case, it is possible to define simple and intuitive tableau systems. We shall present a tableau system for *QHL* which handles equality, and rigid and non-rigid designators.

A lot could technically be said about the tableau systems discussed here, and we mention some results at the end of the paper. Moreover, a lot could (and should) be said about why we feel *QHL* is a good tool for handling many traditional applications of orthodox quantified modal logic, but (apart from an example showing that *QHL* handles validities that Montague-style systems don't) space considerations prevent us from discussing this topic.

2 Quantified Hybrid Logic

We first define the syntax of *QHL*. We have a set NOM of nominals, a set SVAR of state variables, a set FVAR of first-order variables, a set CON of first-order constants, a set IC of unary function symbols, and predicates of any arity (note that predicates of nullary arity are simply propositional variables). The *terms* of the language are the constants from CON, the first-order variables from FVAR and the terms generated by the rule

$$\text{if } q \in \mathsf{IC} \text{ and } s \in \mathsf{NOM} \cup \mathsf{SVAR}, \text{ then } @_s q \text{ is a term.}$$

(For readers familiar with propositional hybrid logic, this notation may come as a surprise: we are combining a satisfaction operator with a term to make a new term. But as the semantics defined below will show, overloading the @ notation in this way is quite natural: $@_s q$ will be the value of the non rigid term q at the world named by s.)

We use (more or less consistently) the following symbols for these different syntactic entities:

Nominals	NOM	n, m, s, t
State variables	SVAR	w, s, t
Propositional variables		$p, r, \ldots$
First-order constants	CON	$c, d, c_i, \ldots$
Non rigid designators	IC	$q, q_i \ldots$
First-order variables	FVAR	x, y, v
First-order terms		$t_i, t_j, \ldots$

The *atomic formulas* are all symbols in NOM and SVAR together with the usual first-order atomic formulas generated from the predicate symbols and equality using the terms. *Complex formulas* are generated from these according to the rules

$$\neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \to \psi \mid \exists x\phi \mid \forall x\phi \mid \Diamond\phi \mid \Box\phi \mid @_n\phi \mid {\downarrow} w.\phi.$$

Here $x \in \mathsf{FVAR}$, $w \in \mathsf{SVAR}$ and $n \in \mathsf{NOM} \cup \mathsf{SVAR}$.

These formulas are interpreted in first-order modal models with constant domains. A *QHL* model is a structure $(W, R, D, I_{nom}, I_{con}, I_w)_{w \in W}$ such that

- (W, R) is a modal frame;
- I_{nom} is a function assigning members of W to nominals in NOM;
- I_{con} is a function assigning elements of D to constants in CON;
- for each $w \in W$, (D, I_w) is an ordinary first-order model.

To interpret formulas with free variables we use special two–sorted assignments. A *QHL assignment* is a function g from $\mathsf{SVAR} \cup \mathsf{FVAR}$ to $W \cup D$ which sends state variables to members of W and first-order variables to elements of D. Given a model and an assignment g, the interpretation of terms t, denoted by $\bar{t}$, is defined as

$$
\begin{array}{lcll}
\bar{x} & = & g(x) & \text{for } x \text{ a variable} \\
\bar{c} & = & I_{con}(c) & \text{for } c \text{ a constant} \\
\overline{@_n q} & = & I_{\mathbf{n}}(q) & \text{for } q \text{ a non rigid designator,} \\
 & & \multicolumn{2}{l}{\text{and } \mathbf{n} \text{ is } I_{nom}(n) \text{ if } n \text{ a nominal, or } g(n) \text{ if } n \text{ a state variable.}}
\end{array}
$$

Formulas are now interpreted as usual. With g^x_d we denote the assignment which is just like g except that $g(x) = d$. $\mathfrak{M}, g, s \Vdash \phi$ means that ϕ holds in model $\mathfrak{M}$ at state s under the assignment g. The inductive definition is as follows:

$$
\begin{array}{lcl}
\mathfrak{M}, g, s \Vdash P(t_1, \dots, t_n) & \iff & \langle \bar{t}_1, \dots, \bar{t}_n \rangle \in I_s(P) \\
\mathfrak{M}, g, s \Vdash t_i = t_j & \iff & \bar{t}_i = \bar{t}_j \\
\mathfrak{M}, g, s \Vdash n & \iff & I_{nom}(n) = s, \text{ for } n \text{ a nominal} \\
\mathfrak{M}, g, s \Vdash w & \iff & g(w) = s, \text{ for } w \text{ a state variable} \\
\mathfrak{M}, g, s \Vdash \neg\phi & \iff & \mathfrak{M}, g, s, \not\Vdash \phi \\
\mathfrak{M}, g, s \Vdash \phi \wedge \psi & \iff & \mathfrak{M}, g, s \Vdash \phi \text{ and } \mathfrak{M}, g, s \Vdash \psi \\
\mathfrak{M}, g, s \Vdash \phi \vee \psi & \iff & \mathfrak{M}, g, s \Vdash \phi \text{ or } \mathfrak{M}, g, s \Vdash \psi \\
\mathfrak{M}, g, s \Vdash \phi \rightarrow \psi & \iff & \mathfrak{M}, g, s \Vdash \phi \text{ implies } \mathfrak{M}, g, s \Vdash \psi \\
\mathfrak{M}, g, s \Vdash \exists x \phi & \iff & \mathfrak{M}, g^x_d, s \Vdash \phi, \text{ for some } d \in D \\
\mathfrak{M}, g, s \Vdash \forall x \phi & \iff & \mathfrak{M}, g^x_d, s \Vdash \phi, \text{ for all } d \in D \\
\mathfrak{M}, g, s \Vdash \Diamond\phi & \iff & \mathfrak{M}, g, t \Vdash \phi \text{ for some } t \in W \text{ such that } Rst \\
\mathfrak{M}, g, s \Vdash \Box\phi & \iff & \mathfrak{M}, g, t \Vdash \phi \text{ for all } t \in W \text{ such that } Rst \\
\mathfrak{M}, g, s \Vdash @_n\phi & \iff & \mathfrak{M}, g, I_{nom}(n) \Vdash \phi \text{ for } n \text{ a nominal} \\
\mathfrak{M}, g, s \Vdash @_w\phi & \iff & \mathfrak{M}, g, g(w) \Vdash \phi \text{ for } w \text{ a state variable} \\
\mathfrak{M}, g, s \Vdash {\downarrow} w.\phi & \iff & \mathfrak{M}, g^w_s, s \Vdash \phi.
\end{array}
$$

3 The tableau calculus

The tableau system can be divided into three natural pieces: **(A)** the propositional rules, the $\Diamond$ and $\Box$ rules and the rules for @; **(B)** the rule for $\downarrow$; **(C)** the rules for (first-order) quantification and equality. The blocks of rules taken separately form a complete calculus for the appropriate reducts. In particular:

1. **A** is complete for the propositional modal language expanded with nominals and @. (We name this system $\mathcal{HL}(@)$; in the literature it is often called the *basic hybrid language.*)

2. $\mathbf{A} \cup \mathbf{B}$ is complete for $\mathcal{HL}(@, \downarrow)$, the expansion of $\mathcal{HL}(@)$ with state variables and the $\downarrow$ binder;

3. $\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}$ is complete for QHL.

Some terminology. As usual, a tableau branch is *closed* if it contains ϕ and $\neg\phi$, where ϕ is a formula. A tableau is closed if each branch is closed. A branch is *atomically closed* if it closes on an atom and its negation. A *(tableau) proof* of a hybrid sentence ϕ is a closed tableau beginning with $\neg @_s\phi$, where s is a nominal not occurring in ϕ.

3.1 Tableau for $\mathcal{HL}(@)$

A key feature of our tableau is that all modal formulas occurring in a proof are grounded to a named world by their label. (This same feature also occurs in labelled tableau for propositional modal logic [7, 6].)

Grounding to a named state is implemented in our system by ensuring that all formulas occurring in proofs are of the form $@_s\phi$ or $\neg @_s\phi$ for s a nominal. Thus the propositional rules become

Conjunctive rules

$$\frac{@_s(\phi \wedge \psi)}{\begin{array}{c}@_s\phi\\ @_s\psi\end{array}} \qquad \frac{\neg @_s(\phi \vee \psi)}{\begin{array}{c}\neg @_s\phi\\ \neg @_s\psi\end{array}} \qquad \frac{\neg @_s(\phi \to \psi)}{\begin{array}{c}@_s\phi\\ \neg @_s\psi\end{array}}$$

Disjunctive rules

$$\frac{@_s(\phi \vee \psi)}{@_s\phi \mid @_s\psi} \qquad \frac{\neg @_s(\phi \wedge \psi)}{\neg @_s\phi \mid \neg @_s\psi} \qquad \frac{@_s(\phi \to \psi)}{\neg @_s\phi \mid @_s\psi}$$

Negation rules

$$\frac{\neg @_s\neg\phi}{@_s\phi} \qquad \frac{@_s\neg\phi}{\neg @_s\phi}$$

To these we add rules for diamond and box. In the diamond rules, t is a nominal which does not occur on the branch.

Diamond rules

$$\frac{@_s\Diamond\phi}{\begin{array}{c}@_s\Diamond t\\ @_t\phi\end{array}} \qquad \frac{\neg @_s\Box\phi}{\begin{array}{c}@_s\Diamond t\\ \neg @_t\phi\end{array}}$$

Box rules

$$\frac{@_s\Box\phi, \quad @_s\Diamond t}{@_t\phi} \qquad \frac{\neg @_s\Diamond\phi, \quad @_s\Diamond t}{\neg @_t\phi}$$

Finally the rules for @. There are two rewrite rules to delete nestings of @. Next, as $@_st$ really means that s and t are equal, there are rules to handle equality. These three rules are direct analogues of the reflexivity and replacement rules in Fitting's first order tableau system [5]. As we will use them often, we gave them separate names.

@ rules

$$\frac{@_s@_t\phi}{@_t\phi} \qquad \frac{\neg @_s@_t\phi}{\neg @_t\phi} \qquad \frac{[s \text{ on the branch}]}{@_ss}\,[\mathsf{Ref}] \qquad \frac{@_st \quad @_s\varphi}{@_t\varphi}\,[\mathsf{Nom}] \qquad \frac{@_st \quad @_r\Diamond s}{@_r\Diamond t}\,[\mathsf{Bridge}]$$

The following rules can be derived using Nom and Ref.

$$\frac{@_st}{@_ts}\,[\mathsf{Sym}] \qquad \frac{@_st \quad @_tr}{@_sr}\,[\mathsf{Trans}] \qquad \frac{@_st \quad @_t\varphi}{@_s\varphi}\,[\mathsf{Nom}^{-1}]$$

Here is a proof for Nom^{-1} which incorporates a proof of Sym. Trans is just an instantiation of Nom^{-1}.

(1)	$@_s t$,	$@_t \phi$	assumption
(2)	$@_s s$		application of Ref
(3)	$@_t s$		by Nom from (1), (2)
(4)	$@_s \phi$		by Nom from (1), (3).

Example. Figure 1 below contains a tableau proof for the validity

(1) $(\Diamond p \wedge \Diamond \neg p) \rightarrow (\Box(q \rightarrow n) \rightarrow \Diamond \neg q)$.

Here n is a nominal and p, q are propositional variables. The formula expresses that if a state has two successors, then if it has at most one q successor, it has at least one $\neg q$ successor. Note that this is not expressible in ordinary modal logic. In ordinary modal logic we cannot put an upper bound on the number of successors.

3.2 Tableau for $\mathcal{HL}(\downarrow, @)$

To obtain a complete tableau system for the expansion of $\mathcal{HL}(@)$ with variables over states and the binder $\downarrow$, we only need to add the following two rewrite rules to the rules for $\mathcal{HL}(@)$:

Downarrow rules

$$\frac{@_s{\downarrow}w.\phi}{@_s\phi[s/w]} \qquad \frac{\neg @_s{\downarrow}w.\phi}{\neg @_s\phi[s/w]}$$

Here $[s/w]$ means substitute s for all free occurrences of w in ϕ. Because s is always a nominal, whence cannot be quantified over, we do not have to worry about accidental bindings.

Examples. Using downarrow and the variables over states we can make formulas which express structural properties of the frame. For instance, ${\downarrow}w.\Diamond w$ holds at a state s if and only if s is reflexive; ${\downarrow}w.\Box\Diamond w$ holds at s if and only if $\forall t(Rst \rightarrow Rts)$ holds. When taken universally, these structural properties correspond to the well known axioms $p \rightarrow \Diamond p$ and $\Diamond\Box p \rightarrow p$, respectively. The easy side of this correspondence can be shown in the logic $\mathcal{HL}(\downarrow, @)$. That is, both these formulas are valid

(2) ${\downarrow}w.\Diamond w \quad \rightarrow \quad (p \rightarrow \Diamond p)$

(3) ${\downarrow}w.\Box\Diamond w \quad \rightarrow \quad (\Diamond\Box p \rightarrow p)$.

We give a tableau proof of the first.

1. $\neg @_s({\downarrow}w.\Diamond w \rightarrow (p \rightarrow \Diamond p))$
2. $@_s{\downarrow}w.\Diamond w$
3. $\neg @_s(p \rightarrow \Diamond p)$
4. $@_s\Diamond s$
5. $@_s p$
6. $\neg @_s\Diamond p$
7. $\neg @_s p$.

In this, 2 and 3 are from 1 by a conjunctive rule; 4 is from 2 by a downarrow rule; 5 and 6 are from 3 by a conjunctive rule; 7 is from 4 and 6 by a box rule. The tableau closes on 5 and 7.

1. $\neg @_s(\Diamond p \wedge \Diamond\neg p \rightarrow (\Box(q \rightarrow n) \rightarrow \Diamond\neg q))$
2. $@_s(\Diamond p \wedge \Diamond\neg p)$
3. $\neg @_s(\Box(q \rightarrow n) \rightarrow \Diamond\neg q)$
4. $@_s\Diamond p$
5. $@_s\Diamond\neg p$
6. $@_s\Box(q \rightarrow n)$
7. $\neg @_s\Diamond\neg q$
8. $@_s\Diamond t$
9. $@_t p$
10. $@_s\Diamond r$
11. $@_r\neg p$
12. $@_t(q \rightarrow n)$

13.1 $\neg @_t q$	14. $@_t n$	
13.2 $\neg @_t\neg q$	15. $@_r(q \rightarrow n)$	
13.3 $@_t q$		
	16.1 $\neg @_r q$	17. $@_r n$
	16.2 $\neg @_r\neg q$	18. $@_n t$
	16.3 $@_r q$	19. $@_n r$
		20. $@_t r$
		21. $@_r p$

In this, 2 and 3 are from 1 by a conjunctive rule; 4,5,6,7 are from 2 and 3 by conjunctive rules; 8,9,10,11 are from 4 and 5 by diamond rules; 12 is from 6 and 8 by box; 13.1 and 14 are from 12 by a disjunctive rule; 13.2 is from 7 and 8 by box; 13.3 is from 13.2 by a negation rule. The branch closes on 13.3 and 13.1.
15 is from 6 and 10 by box; 16.1 and 17 are from 15 by a disjunctive rule; 16.2 is from 10 and 7 by box; 16.3 is from 16.2 by a negation rule. The branch closes on 16.1 and 16.3.
18 is from 14 by the derived Sym rule; 19 is from 17 by Sym; 20 is from 18 and 19 by Nom; 21 is from 20 and 9 by the Nom rule. The final branch closes on 21 and 11.

Figure 1: Tableau for $\Diamond p \wedge \Diamond\neg p \rightarrow (\Box(q \rightarrow n) \rightarrow \Diamond\neg q)$.

3.3 Tableau for *QHL*

A complete tableau system for quantified hybrid logic consists of the $\mathcal{HL}(\downarrow, @)$ system, plus the (adjusted) rules for the quantifiers and equality from Fitting's system (see [5]) for first-order logic with equality, plus two rules relating equalities across worlds. In the existential rules, c is a parameter which is new to the branch. As parameters are never quantified over, the substitution $[c/x]$ is free for the formula $\phi(x)$. In the universal rules, t is any grounded term on the branch (thus either a first-order constant, a parameter or a grounded definite description). A grounded definite description is a term $@_n q$ for n a nominal and q a non–rigid designator from IC.

Existential rules

$$\frac{@_s \exists x \phi(x)}{@_s \phi(c)} \qquad \frac{\neg @_s \forall x \phi(x)}{\neg @_s \phi(c)}$$

Universal rules

$$\frac{@_s \forall x \phi(x)}{@_s \phi(t)} \qquad \frac{\neg @_s \exists x \phi(x)}{\neg @_s \phi(t)}$$

Since we assume constant domains, we expect the Barcan formula and its converse to hold. In hybrid logic, they can be formulated in two ways,

$$\Box \forall x \phi \leftrightarrow \forall x \Box \phi \text{ and } @_s \forall x \phi \leftrightarrow \forall x @_s \phi.$$

Both are valid. We present a tableau proof of $\Box \forall x \phi(x) \rightarrow \forall x \Box \phi(x)$.

1. $\neg @_s (\Box \forall x \phi(x) \rightarrow \forall x \Box \phi(x))$
2. $@_s \Box \forall x \phi(x)$
3. $\neg @_s \forall x \Box \phi(x)$
4. $\neg @_s \Box \phi(c)$
5. $@_s \Diamond t$
6. $\neg @_t \phi(c)$
7. $@_t \forall x \phi(x)$
8. $@_t \phi(c)$.

In this, 2 and 3 are from 1 by a conjunctive rule; 4 is from 3 by an existential rule; 5 and 6 are from 4 by a diamond rule; 7 is from 2 and 5 by a box rule; 8 is from 7 by universal instantiation. The tableau closes on 8 and 6.

Besides Fitting's [5] Reflexivity (Ref) and Replacement (RR) rules, there are three extra rules for equality. The first (called DD) states that if n and m denote the same state, then $@_n q$ and $@_m q$ denote the same individual. The second and third (both called $\mathsf{@=}$) embody that equality is a rigid predicate: if two terms are the same in one world, they are the same in every world. Because these two rules peel the leading $@_n$ off equalities, reflexivity and replacement can be kept in the old format.

QHL Equality rules

$$\frac{}{t = t}[\mathsf{Ref}] \qquad \frac{t = u, \quad \phi(t)}{\phi[u]}[\mathsf{RR}], \qquad \frac{@_n m}{@_n q = @_m q}[\mathsf{DD}] \qquad \frac{@_n (t_i = t_j)}{t_i = t_j}[\mathsf{@=}] \qquad \frac{\neg @_n (t_i = t_j)}{\neg (t_i = t_j)}[\mathsf{@=}]$$

In the Replacement rule, $\phi[u]$ denotes $\phi(t)$ with some of the occurrences of t replaced by u.

Examples. The most interesting examples deal with equality. First we show that equality is necessary: $\forall x \forall y(x = y \rightarrow \Box(x = y))$.

The tableau proof:

1. $\neg @_s \forall x \forall y(x = y \rightarrow \Box(x = y))$
2. $\neg @_s(c_1 = c_2 \rightarrow \Box(c_1 = c_2))$
3. $@_s(c_1 = c_2)$
4. $\neg @_s \Box(c_1 = c_2)$
5. $@_s \Diamond t$
6. $\neg @_t(c_1 = c_2)$
7. $c_1 = c_2$
8. $\neg(c_1 = c_2)$.

In this, 2 is from 1 by two applications of an existential rule; 3 and 4 are from 2 by a conjunctive rule; 5 and 6 are from 4 by a diamond rule; 7 and 8 are from 3 and 6 respectively by the two @= rules.

The next example is about rigid and non rigid designators. Consider the sentence *Caroline is Miss America.* When formalising this let c be a rigid designator denoting Caroline and q a non–rigid designator denoting Miss America. Then $\downarrow x.(c = @_x q)$ means *Caroline is the present Miss America.* It is true in a state w if $I_{con}(c) = I_w(q)$. This formula has the following relation with the $\Box$ operator:

(4) $\not\models (\downarrow w.c = @_w q) \quad \rightarrow \quad \Box \downarrow w.c = @_w q$

(5) $\models (\downarrow w.c = @_w q) \quad \rightarrow \quad \downarrow w.\Box c = @_w q.$

A falsifying model for the sentence in (4) is given by two worlds n and m, with Rnm, and a domain $\{a, b\}$ with the interpretation $I_{con}(c) = I_n(q) = a$ and $I_m(q) = b$. Then (4) fails at world n. When downarrow has wide scope in the consequent, the formula becomes true. Here is the tableau proof:

1. $\neg @_n((\downarrow w.c = @_w q) \rightarrow \downarrow w.\Box(c = @_w q))$
2. $@_n \downarrow w.c = @_w q$
3. $\neg @_n \downarrow w.\Box(c = @_w q)$
4. $@_n(c = @_n q)$
5. $\neg @_n \Box(c = @_n q)$
6. $@_n \Diamond m$
7. $\neg @_m(c = @_n q)$
8. $c = @_n q$
9. $\neg(c = @_n q)$.

In this, 2 and 3 are from 1 by a conjunctive rule; 4 and 5 are from 2 and 3 by a downarrow rule, respectively; 6 and 7 are from 5 by a diamond rule; 8 and 9 are from 4 and 7 by an @= rule, respectively.

Finally we give two examples of formulas which are valid in *QHL* but not in Montague's intensional logic IL [8]:

(6) $\forall x \exists y \Box(x = y) \quad \rightarrow \quad \exists y \Box(c = y)$

(7) $c_1 = c_2 \quad \rightarrow \quad (\Box(c_1 = c_1) \rightarrow \Box(c_1 = c_2)).$

The validities are easy to prove by tableau. For the universal instantiation law (6), applying once the universal tableau rule leads to a contradiction:

1. $\neg @_s(\forall x \exists y \Box (x = y) \rightarrow \exists y \Box (c = y))$
2. $@_s(\forall x \exists y \Box (x = y))$
3. $\neg @_s(\exists y \Box (c = y))$
4. $@_s \exists y \Box (c = y)$.

To prove (7) (substitutes can be substituted for equals), the @= rules are needed.

0. $\neg @_s(c_1 = c_2 \rightarrow (\Box (c_1 = c_1) \rightarrow \Box (c_1 = c_2)))$
1. $@_s(c_1 = c_2)$
2. $@_s \Box (c_1 = c_1)$
3. $\neg @_s \Box (c_1 = c_2)$
4. $@_s \Diamond t$
5. $\neg @_t(c_1 = c_2)$
6. $\neg (c_1 = c_2)$
7. $c_1 = c_2$.

In this, 1,2,3 are from 0 by two applications of the conjunctive rule; 4 and 5 are from 3 by a diamond rule; 6 is from 5 by a @= rule; 7 is from 1 by a @= rule. Note that the premise $\Box(c_1 = c_1)$ is not even used.

4 Conclusions

To conclude, some technical remarks. The soundness of the tableau system is clear, but what about completeness? In fact, the system is complete, and the easiest way to prove this is to lift the standard model construction for the basic hybrid language (discussed in detail in Chapter 7 of [3]) to *QHL*.

But the calculus is not only complete, it is also flexible. For a start, it permits us to straightforwardly handle all four standard domain conditions. We presented it only for the constant domain semantics, but slight changes render it complete for varying, expanding or contracting domains. Consider the case of contracting domains. This condition is *not* definable in orthodox first-order modal logic, but it is definable in *QHL*, namely by $\downarrow w.\Box \forall x @_w \exists y (y = x)$. By freely introducing instances of this sentence in the course of tableau proofs, we obtain a system complete with respect to contracting domains.

Furthermore, the fundamental completeness result for propositional hybrid logic extends to the first-order case (irrespective of whether constant, varying, expanding, or contracting domains are assumed). The result is: every first-order definable class of frames which is closed under and reflects generated subframes can be completely axiomatised by adding a pure axiom (see [1]). This result covers many frame classes definable in orthodox modal logic (for example, transitivity, reflexivity and symmetry) and also many others not so definable (for example, irreflexivity, antisymmetry, and discreteness). Thus complete tableau systems for all these frame classes can be obtained by adjoining extra (pure) axioms to the tableau system just presented, so we automatically have complete tableau systems for hybrid analogs of **T**, **D**, **B**, **S4**, and **S5**, and for a countable infinity of other logics besides.

Finally, in an unpublished companion paper we have shown that the presented calculus can be used to construct interpolants. That interpolation holds for *QHL* and all pure axiomatic extensions was proved in [2] by a model theoretic argument. But it is useful to have a mechanism for explicitly calculating interpolants, and the tableau system presented in this paper lets us do precisely that.

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