Internalization: The Case of Hybrid Logics

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Abstract

A sequent calculus for hybrid logics is developed from a calculus for classical predicat logic by a series of transformations. We formalize the semantic theory of hybrid logic using a sequent calculus for predicate logic plus axioms. This works, but it is ugly. The unnattractive features are removed one-by-one, until the final vestiges of the metalanguage can be set aside to reveal a fully internalized calculus. The techniques are quite general and can be applied to a wide range of hybrid and modal logics.

Keywords: Proof theory, cut-elimination, internalization, hybrid logic, modal loigc, subformula property, sequent calculus.

1 Introduction

The classical beauty of Gentzen's sequent calculus is obvious from first acquaintance. Each logical operator is precisely characterized by a pair of rules with perfect economy. The operators appear only in the conclusion of the rules, which are constructed from a tidy arrangement of subformulas. Every symbol occurs only in the place that best explains its function; nothing is wasted.

The fit between the geometry of sequent proofs and classical predicate logic is almost too perfect. When we try to use similar techniques with other logics, it never quite works. With intuitionistic logic the left-right symmetry is broken; with modal logics, there are ugly restrictions on the contexts. This paper is a contribution to our understanding of why this happens.

We propose a means of analyzing the form of a logic by the process of internalization. A calculus is *fully internalized* if the only symbols that occur in the rules of the calculus are symbols of the object language. No labels or special positional operators are allowed—only the geometry of syntactic structure and rules governing how logical symbols may be added and subtracted. Gentzen's sequent calculi are all fully internalized, as are many of the calculi proposed for modern applied logics. Yet almost all of them require the symmetry-breaking contortions of intuitionistic and modal logic.

Contrast the situation in proof theory with that in model theory. There, applied logics have a natural home in the world of Kripke structures. A semantic theory of relations can be constructed for almost all known logics, and each one is more-or-less as good as the others. The theory proposes a network of extra-logical machinery—accessibility relations and the like—in terms of which the logical operators are translated. This is an *external* approach to logic because it uses much beyond the syntax of the object language.

The passage from external semantic theory to fully internalized calculus is well-understood, especially among those logicians who can knock up a sequent calculus for a new modal logic before breakfast. The strategy involves an implicit translation of everything back into first-order predicate logic. The axioms of the semantic theory are typically first-order, and so their logical properties can be seen through their representation in the sequent calculus for classical first-order predicate logic. The trick is to see how to get the effect of the classical sequent rules using the syntax of the new logic.

In this paper, we formalize the process of internalization explicitly using, first, an expression of the external semantic theory in classical predicate logic, and then a series of transformations taking us to more-and-more internalized calculi, dropping the metalogical props one by one.

To illustrate the process, we focus on hybrid logics. Hybrid logics lie at the boundary between predicate logic and propositional modal logic, making them an especially appropriate focus for proof-theoretic techniques that also cross between these realms. Recent papers [3, 16, 8, 7] have developed a number of proof systems for hybrid logics, some of which are fully internalized. And, of course, they all have clearly formulated Kripke-style semantic theories. They form a perfect case study for a theory of internalization![1]

We begin Section 2 by reviewing the semantic theory of hybrid logics, which we express as a set of axioms in a formal metalanguage. In Section 3 we introduce a variant of the classical sequent calculus for predicate logic that is especially good at handling equality, and add axioms to capture the formal semantics of hybrid logics. Section 4 regains the Subformula Property, which was lost with the addition of axioms, by generating rules for the hybrid operators. The Subformula Property is essential to the process because it allows us to throw away extraneous rules while maintaining completeness. The resulting system still uses metalogical labels to control the flow of information in a proof, but these are removed with the use of the hybrid operators in Section 5.

2 Hybrid Languages and their Formal Semantics

Hybrid logics have a long history and a number of interesting applications. We refer the reader to [10] for a wealth of information about these logics and an extensive bibliography. Our present interest in hybrid logics is merely that of providing an example of a broad class of logics for which the internalization strategy is particularly appropriate.

[1] Naturally, the fit is too good to be a matter of chance. My awareness of the process of internalization came from earlier attempts at providing a sequent calculus for hybrid logics. These are documented in [3].

There are two routes to understanding hybrid logics. The first, and most common, is to think of them as propositional modal logics enriched with various devices from predicate logic to increase their expressive power. An alternative perspective, adopted here, sees them as the contextualization of predicate logic.

Let L be the language of predicate logic with individual variables x $(= x_1, x_2, \ldots)$, property symbols p $(= p_1, p_2, \ldots)$, and (binary) relation symbols r $(= r_1, r_2, \ldots)$. Closed formulas of L express properties of relational structures via the definition of truth in a structure. For example, the formula $\forall x\, rxx$ is satisfied by those structures having a reflexive r-relation.

Hybrid logic results from the contextualization of these properties to elements of a structure. We aim to express the contextual properties of an element of the structure using formulas with an implicit parameter that refers to that element. Any formula of L with one free variable expresses such a property—we move to hybrid logic by erasing the free variable and assuming that its reference is supplied as an implicit parameter.

A property symbol p combines with a variable x to give a formula px. Hide the x and you get the hybrid formula p, which can be combined with other formulas using the standard Boolean operators. The formula $\exists y\, (rxy \,\&\, py)$ becomes a modal formula $\Diamond_r p$ when the variable x is hidden and assumed to refer to the point of evaluation. All this is present in ordinary modal logic. We get the specifically hybrid operators by applying the same idea to a wider class of formulas. For example, when the x of $x = y$ is hidden we get the nominal y, but now as a formula that can be combined with other formulas using logical operators. In a context in which y refers to an element a of a structure, the new hybrid formulas y expresses the property of being identical to a.

The language H of hybrid logic is defined as follows. The atomic formulas of H are the individual variables and the property symbols. Complex formulas are built up using the Boolean operators $\vee$ and $\sim$ together with unary modal operators $\Diamond$,$\Diamond_r$, and $\exists x$. Negation duals of the operators, &, $\Box$, $\Box_r$, and $\forall x$ are defined as abbreviations. The distinction between free and bound variables applies to H just as it does to L. The semantics of H is a straightforward adaption of the semantics of L. An interpretation for H is an interpretation for L, namely a relational structure $\mathbf{A}$ of type $(p_1, p_2, \ldots, r_1, r_2, \ldots)$ with a distinguished point, a in the domain $|\mathbf{A}|$ of $\mathbf{A}$[2]. Given an assignment g of elements of $|\mathbf{A}|$ to the variables, we define the relation $\models$ (*satisfies*) as follows:

$\mathbf{A}, a, g \models x$	if $g(x) = a$
$\mathbf{A}, a, g \models p$	if $a \in p^{\mathbf{A}}$
$\mathbf{A}, a, g \models \sim\varphi$	if not $\mathbf{A}, a, g \models \varphi$
$\mathbf{A}, a, g \models (\varphi \vee \psi)$	if either $\mathbf{A}, a, g \models \varphi$ or $\mathbf{A}, a, g \models \psi$
$\mathbf{A}, a, g \models \Diamond\varphi$	if for some $a' \in |\mathbf{A}|$, $\mathbf{A}, a', g \models \varphi$
$\mathbf{A}, a, g \models \Diamond_r\varphi$	if for some $a' \in |\mathbf{A}|$, $\langle a, a'\rangle \in r^{\mathbf{A}}$ and $\mathbf{A}, a', g \models \varphi$
$\mathbf{A}, a, g \models \exists x\, \varphi$	if for some $a' \in |\mathbf{A}|$, $\mathbf{A}, a, g[^x_{a'}] \models \varphi$

where, as usual, $g[^x_{a'}](y)=a'$ if $y = x$, and $g(y)$ otherwise.

[2]A *relational structure* of type $(p_1, p_2, \ldots, r_1, r_2, \ldots)$ is a set $|\mathbf{A}|$ together with a subset $p_i^{\mathbf{A}}$ of $|\mathbf{A}|$ for each p_i and a subset $r_i^{\mathbf{A}}$ of $|\mathbf{A}|^2$ for each r_i.

This is immediately recognizable as containing the Kripke semantics for a language of modal logic in which p is a propositional variable and $\Diamond$ is the S5 modal operator. $\Diamond_r$ is also a normal modal operator, interpreted using the accessibility relation $r^{\mathbf{A}}$. What has been added is distinctly hybrid: individual variables occurring as formulas (*nominals*) and the modal quantifier $\exists x$

Sequents of the languages mentioned above are expressions of the form $\Gamma \longrightarrow \Delta$, in which Γ and Δ are lists of formulas. Such a sequent is *valid* if every $\mathbf{A}$, a,g satisfying every formula in Γ also satisfies some formula in Δ.

The hybrid character of the language is further developed with the operators $@_x$ and $\downarrow_x$. The only syntactic difference between these is that the *downarrow* operator $\downarrow_x$ binds its variable, whereas the *at* operator, $@_x$, does not. They are interpreted in a relational structure as follows:

$$\begin{array}{ll} \mathbf{A}, a, g \models @_x\varphi & \text{if } \mathbf{A}, g(x), g \models \varphi \\ \mathbf{A}, a, g \models \downarrow_x \varphi & \text{if } \mathbf{A}, a, g[^x_a] \models \varphi \end{array}$$

Clearly, there are many other possibilities for hybrid operators, and yet there is already a kind of expressive completeness. The above two operators are definable in H as $\Diamond(x \,\&\, \varphi)$ and $\exists x\,(x \,\&\, \varphi)$. In fact, every operator definable in the predicate language L is also definable in H (see [4]).

It may seem a little strange to focus on such a richly expressive language as H. Much of the research on hybrid languages (for example, [5, 6, 2, 1]) has concentrated on fragments of H with much less expressive power and correspondingly lower computational complexity. We ensure that the rules developed for H can be applied to fragments by insisting that they have a subformula property. On the strictest interpretation, a rule is said to have the Subformula Property if every formula occurring in an application of a rule is a subformula of the conclusion. A complete calculus in which no rule involves more than one operator and every rule has the Subformula Property is guaranteed to be *modular*: select any set of operators and the set of rules using those operators will be complete for the fragment of the language formed from those operators. Similar results can be obtained even when the Subformula Property is weakened to allow a restricted class of formulas for each rule, so long as the fragment incudes the formulas in the restricted classes.

We are now ready to formalize the semantic theory of H. Let M (for 'meta') be the language of predicate logic extending L with new atomic formulas of the form $x{:}\,\varphi$, where φ is a formula of H. The free variables of $x{:}\,\varphi$ are x together with the free variables of φ. To make it (conceptually) clear that M is a language of predicate logic, we should have introduced a fresh $n+1$-ary relation symbol for each formula φ of H with n free variables. But this would have made the formulas much less clear (to read) and so we allow what is obviously a merely notational variant.

The semantics of H can be expressed directly in M as follows:

$$\begin{array}{ll} \forall z\,\forall x\,(z{:}\,x \equiv x = z) & \forall z\,(z{:}\,\Diamond\varphi \equiv \exists x\, x{:}\,\varphi) \\ \forall z\,(z{:}\,p \equiv pz) & \forall z\,(z{:}\,\Diamond_r\varphi \equiv \exists x\,(rzx \;\&\; x{:}\,\varphi)) \\ \forall z\,(z{:}\,{\sim}\varphi \equiv {\sim}z{:}\,\varphi) & \forall z\,\forall x\,(z{:}\,@_x\varphi \equiv x{:}\,\varphi) \\ \forall z\,(z{:}\,(\varphi \vee \psi) \equiv (z{:}\,\varphi \vee z{:}\,\psi)) & \forall z\,(z{:}\,\downarrow_x \varphi \equiv z{:}\,\varphi[^x_z]) \\ \forall z\,(z{:}\,\exists x\,\varphi \equiv \exists x\, z{:}\,\varphi) & \end{array}$$

Similar sentences could be produced for any other first-order definable hybrid operator. Let Θ be the above set of sentences of M. A sequent $\Gamma \longrightarrow \Delta$ is Θ-*valid* if for every model $\mathbf{A}$ of Θ and every assignment g, if $\mathbf{A}, g$ satisfies all the formulas in Γ, then it also satisfies some of the formulas in Δ. From such a characterization of the semantics of H, we get the following lemma. For each list Γ of formulas of H, let $u{:}\,\Gamma$ be the list of formulas $u{:}\,\varphi$, for each φ in Γ.

Lemma 1 A sequent $\Gamma \longrightarrow \Delta$ of H is valid if and only if $u{:}\,\Gamma \longrightarrow u{:}\,\Delta$ is Θ-valid.

Proof It is enough to observe that there is a one-one correspondence between models $\mathbf{B}$ of Θ and relational structures $\mathbf{A}$ for H, such that $|\mathbf{A}|{=}|\mathbf{B}|$ and for every assignment g, element a of the domain, formula φ of H, and variable x not in φ,

$$\mathbf{A}, a, g \models \varphi \text{ if and only if } \mathbf{B}, g[^x_a] \models x{:}\,\varphi$$

For any Θ-model $\mathbf{B}$, the corresponding structure $\mathbf{A}$ is just the reduct to properties $p_1, p_2, \ldots$ and relations $r_1, r_2, \ldots$. The correspondence is one-one because the theory Θ fixes the interpretation of all of the new relation symbols.

3 Sequent Calculus for Θ-validity

Valid sequents in M can be generated in a uniform way using the sequent calculus SM shown in Figure 1. Here φ and ψ range over formulas of M and Γ and Δ range over lists of formulas of M. We write $\Gamma \approx \Delta$ to mean that Γ and Δ contain the same *set* of formulas. For any expression σ, we write $\sigma[^x_y]$ for the result of replacing all free occurrences of x in σ by y.

The rules are written in *horizontal notation*, with premises listed to the left of '$\Rightarrow$' and conclusion to the right[3].

For proof-theoretic purposes it is useful to make a global decision about which variables can occur free in a sequent. Let $u_1, u_2, \ldots$ be a distinguished class of individual variables, called *parameters*. We restrict our attention to sequents whose free variables are all parameters and whose bound variables are not parameters, so that awkward clashes of variables can be avoided. We use u, v, w to range over parameters and x, y, z to range over the more inclusive class of individual variables.
The sequent to the right of the arrow $\Rightarrow$ is the *conclusion* of the rule; those on the left are its *premises*. A sequent $\Gamma \longrightarrow \Delta$ is a *theorem* of a set of rules, such as SM, if it is generated by them. In other words, a sequent is a theorem if it is the conclusion of a rule whose premises (if it has premises) are also theorems. The generation tree for a theorem is called a *proof*[4]. The calculus SM has the celebrated *Subformula Property*: every formula occurring in a proof of $\Gamma \longrightarrow \Delta$ is a subformula of a formula in Γ or Δ. The rule of *Weakening* (W),

$$\Gamma \longrightarrow \Delta \quad \Rightarrow \quad \Gamma', \Gamma \longrightarrow \Delta, \Delta'$$

[3]This is equivalent to *vertical notation*, often seen in textbooks, in which premises and conclusion are separated by a deduction line of the kind used in proofs. I prefer the horizontal notation because it is less cumbersome and makes a clearer distinction between the *rules* of a system and the *proofs* produced when the rules are applied.

[4]Note that when we draw a proof as a tree, any application of S will not be shown unless it is especially significant.

Structural Rules		
I		$\Rightarrow \quad \varphi, \Gamma \longrightarrow \Delta, \varphi.$
S	$\Gamma \longrightarrow \Delta$	$\Rightarrow \quad \Gamma' \longrightarrow \Delta'$ if $\Gamma \approx \Gamma'$ and $\Delta \approx \Delta'$.
Logical Rules		
$\sim$L	$\Gamma \longrightarrow \Delta, \varphi$	$\Rightarrow \quad \sim\varphi, \Gamma \longrightarrow \Delta.$
$\sim$R	$\varphi, \Gamma \longrightarrow \Delta$	$\Rightarrow \quad \Gamma \longrightarrow \Delta, \sim\varphi.$
$\vee$L	$\varphi, \Gamma \longrightarrow \Delta;\ \psi, \Gamma \longrightarrow \Delta$	$\Rightarrow \quad (\varphi \vee \psi), \Gamma \longrightarrow \Delta$
$\vee$R	$\Gamma \longrightarrow \Delta, \varphi, \psi$	$\Rightarrow \quad \Gamma \longrightarrow \Delta, (\varphi \vee \psi)$
$\exists$L	$\varphi[^x_u], \Gamma \longrightarrow \Delta$	$\Rightarrow \quad \exists x\, \varphi, \Gamma \longrightarrow \Delta$ if u does not occur in φ, Γ, Δ.
$\exists$R	$\Gamma \longrightarrow \Delta, \varphi[^x_u]$	$\Rightarrow \quad \Gamma \longrightarrow \Delta, \exists x\, \varphi$
$=$L$_1$	$u = v, \Gamma[^w_u] \longrightarrow \Delta[^w_u]$	$\Rightarrow \quad u = v, \Gamma[^w_v] \longrightarrow \Delta[^w_v].$
$=$L$_2$	$u = v, \Gamma[^w_v] \longrightarrow \Delta[^w_v]$	$\Rightarrow \quad u = v, \Gamma[^w_u] \longrightarrow \Delta[^w_u].$
$=$R		$\Rightarrow \quad \Gamma \longrightarrow \Delta, u = u$

Figure 1: The sequent calculus SM

is *admissible*, which means that its addition to the set of rules will not allow us to prove more theorems.
Of the rules of SM, only the Barwise equality rules, $=$L$_1$ and $=$L$_2$, may be unfamiliar[5]. Together they allow the replacement of any number of occurrences of u by v and v by u in a sequent containing $u = v$ on the left side. An advantage of the Barwise rules is the straightforward way in which the following result may be stated:

Theorem 2 The rule of *Cut* (C)

$$\Gamma \longrightarrow \Delta, \varphi;\ \varphi, \Gamma' \longrightarrow \Delta' \quad \Rightarrow \quad \Gamma, \Gamma' \longrightarrow \Delta, \Delta'$$

is admissible in SM.

Proof The theorem is proved by the method of cut-elimination: we show that every application of C—called a *cut*—can be pushed up the proof tree until it falls off the leaves. Technically, this is done by assigning a number to each cut—its *cut rank*—and transforming the proof so as to reduce distance between the cuts of maximal rank and the leaves. Cuts at the leaves are shown to be replaceable by axioms (I or $=$R, in this case). The transformations are of two kinds. In the primary case, the cut formula is assumed to be the principal formula of both rules immediate above the cut. Typically, the transformation replaces the cut with one or more cuts of lower rank. For example, if the cut formula is $\sim\varphi$ we have

$$\dfrac{\dfrac{\overset{\pi_1}{\varphi, \Gamma \longrightarrow \Delta}}{\Gamma \longrightarrow \Delta, \sim\varphi}\,\sim\mathrm{R} \qquad \dfrac{\overset{\pi_2}{\Gamma' \longrightarrow \Delta', \varphi}}{\sim\varphi, \Gamma' \longrightarrow \Delta'}\,\sim\mathrm{L}}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\mathrm{C} \quad \rightsquigarrow \quad \dfrac{\overset{\pi_2}{\Gamma' \longrightarrow \Delta', \varphi} \qquad \overset{\pi_1}{\varphi, \Gamma \longrightarrow \Delta}}{\Gamma', \Gamma \longrightarrow \Delta', \Delta}\,\mathrm{C}$$

[5]They were used by Jon Barwise in some early work on infinitary logic.

The cut rank of φ is less than that of $\sim\varphi$, so this is an improvement. The second kind of transformation occurs when the cut formula is not the principal formula of the final rule of one of the two branches. In this case, we must show how the cut can be moved up that branch, closer to the leaf. For example,

$$\dfrac{\dfrac{\overset{\pi_1}{\varphi, \Gamma \longrightarrow \Delta, \exists x\,\psi}}{\Gamma \longrightarrow \Delta, \sim\varphi, \exists x\,\psi}\,{\sim}\mathrm{R} \qquad \overset{\pi_2}{\exists x\,\psi, \Gamma' \longrightarrow \Delta'}}{\Gamma, \Gamma' \longrightarrow \Delta, \sim\varphi, \Delta'}\,\mathrm{C} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\pi_1}{\varphi, \Gamma \longrightarrow \Delta, \exists x\,\psi} \qquad \overset{\pi_2}{\exists x\,\psi, \Gamma' \longrightarrow \Delta'}}{\varphi, \Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\mathrm{C}}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta', \sim\varphi}\,{\sim}\mathrm{R}$$

For the system without the equality rules, the transformations are all standard (see, for example, [15]). The treatment of the equality rules deserves some comment. In the primary case, there is no difficulty at all in removing cuts whose cut formula is an equation:

$$\dfrac{\dfrac{}{\Gamma \longrightarrow \Delta, u = u}\,{=}\mathrm{R} \qquad \dfrac{\overset{\pi}{u = u, \Gamma'[^{w}_{u}] \longrightarrow \Delta'[^{w}_{u}]}}{u = u, \Gamma'[^{w}_{u}] \longrightarrow \Delta'[^{w}_{u}]}\,{=}\mathrm{L}_1}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\mathrm{C}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{}{\Gamma \longrightarrow \Delta, u = u}\,{=}\mathrm{R} \qquad \overset{\pi}{u = u, \Gamma'[^{w}_{u}] \longrightarrow \Delta'[^{w}_{u}]}}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\mathrm{C}$$

But the secondary case is a little more involved:

$$\dfrac{\dfrac{\overset{\pi_1}{u = v, \Gamma[^{w}_{u}] \longrightarrow \Delta[^{w}_{u}], \varphi[^{w}_{u}]}}{u = v, \Gamma[^{w}_{v}] \longrightarrow \Delta[^{w}_{v}], \varphi[^{w}_{v}]}\,{=}\mathrm{L}_1 \qquad \overset{\pi_2}{\varphi[^{w}_{v}], \Gamma' \longrightarrow \Delta'}}{u = v, \Gamma[^{w}_{v}], \Gamma' \longrightarrow \Delta[^{w}_{v}], \Delta'}\,\mathrm{C}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\pi_1^*[^{v}_{u}]}{u = u, \Gamma[^{w}_{u}][^{v}_{u}] \longrightarrow \Delta[^{w}_{u}][^{v}_{u}], \varphi[^{w}_{u}][^{v}_{u}]} \qquad \overset{\pi_2^*[^{v}_{u}]}{\varphi[^{w}_{v}][^{v}_{u}], \Gamma'[^{v}_{u}] \longrightarrow \Delta'[^{v}_{u}]}}{u = u, \Gamma[^{w}_{u}][^{v}_{u}], \Gamma'[^{v}_{u}] \longrightarrow \Delta[^{w}_{u}][^{v}_{u}], \Delta'[^{v}_{u}]}\,\mathrm{C}}{u = v, u = u, \Gamma[^{w}_{u}][^{v}_{u}], \Gamma'[^{v}_{u}] \longrightarrow \Delta[^{w}_{u}][^{v}_{u}], \Delta'[^{v}_{u}]}\,\mathrm{W}}{u = v, u = v, \Gamma[^{w}_{u}], \Gamma' \longrightarrow \Delta[^{w}_{u}], \Delta'}\,{=}\mathrm{L}_2$$

The problem here is that the cut formula $\varphi[^{w}_{v}]$ becomes $\varphi[^{w}_{u}]$ on the left side, but stays as $\varphi[^{w}_{v}]$ on the right side. The solution is to replace v by u throughout the two branches, after first changing any parameters introduced by the restricted quantifier rule $\exists$L[6]. This does not increase the length of the branches. The cut can be moved up the left branch because $\varphi[^{w}_{u}][^{v}_{u}]$ is identical to $\varphi[^{w}_{v}][^{v}_{u}]$. After the new cut, the vs can be put back in their proper places using $=$L$_2$[7].

[6] We write π_i^* for the result of renaming the parameters in π_i that are introduced by a restricted rule, such as $\exists$L.

[7] In the manuscript, [14], cut-elimination is proved in a less direct manner, using redundant equality axioms in addition to the rules. In effect, the substitution we perform above is there postponed until the elimination algorithm reaches the leaves of the tree.

Theorem 3 A sequent of M is valid if and only if it is a theorem of SM.

Proof To demonstrate soundness (the 'if' direction), we need only observe that if the premises of an SM rule are valid, then so is the conclusion. Completeness (the 'only if' direction) follows from Theorem 2 and the completeness of the predicate calculus with identity, given the derivability of standard sequent axioms of predicate logic[8]. Since the only non-standard rules are the equality rules, we need only observe that reflexivity, symmetry, and transitivity are easily derived:

$$\dfrac{}{\longrightarrow u = u}{=}\mathrm{R} \qquad \dfrac{\dfrac{}{u = v \longrightarrow u = u}{=}\mathrm{R}}{u = v \longrightarrow v = u}{=}\mathrm{L}_2 \qquad \dfrac{\dfrac{\dfrac{}{u = v, v = w \longrightarrow u = u}{=}\mathrm{R}}{u = v, v = w \longrightarrow u = v}{=}\mathrm{L}_2}{u = v, v = w \longrightarrow u = w}{=}\mathrm{L}_2$$

The Subformula Property allows us to extend this result to all fragments of M obtained by removing logical operators.

Corollary 4 For any fragment M' of M that is closed under subformulas—i.e., all subformulas of formulas in M' are also in M'—let SM' be the set of rules of SM that involve operators occurring in M', together with the structural rules. Then a sequent of M' is valid if and only if it is a theorem of SM'.

Proof Soundness (the 'if' direction) follows from the soundness of SM. For the converse, suppose that $\Gamma \longrightarrow \Delta$ is a valid sequent of M'. Then, by Theorem 3 it is a theorem of SM. By the Subformula Property, its proof contains only formulas in M'. The rules used in the proof are therefore in SM', and so $\Gamma \longrightarrow \Delta$ is a theorem of SM'.

These results are easily strengthened to deal with Θ-validity.

Corollary 5 A sequent $\Gamma \longrightarrow \Delta$ of M is Θ-valid if and only if there are formulas $\varphi_1, \ldots, \varphi_n$ in Θ such that $\varphi_1, \ldots, \varphi_n, \Gamma \longrightarrow \Delta$ is a theorem of SM.

Proof By the compactness of predicate logic, and Theorem 3.

The simplest way of extending SM to a calculus for Θ-validity, is to add a new axiom $\Gamma \quad \Rightarrow \quad \Delta, \varphi$ for each sentence φ in Θ. Call the set of these new axioms AΘ.

Theorem 6 A sequent of M is a theorem of S(M+AΘ+C) if and only if it is Θ-valid.

Proof The new rules are obviously Θ-valid, and all the old rules preserve validity and hence also Θ-validity. So, every theorem of S(M+AΘ+C) is Θ-valid. Conversely, suppose that $\Gamma \longrightarrow \Delta$ is Θ-valid. By Corollary 5, there are formulas $\varphi_1, \ldots, \varphi_n$ in Θ, such that $\varphi_1, \ldots, \varphi_n, \Gamma \longrightarrow \Delta$ is a theorem of SM. But $\longrightarrow \varphi_i$ is in AΘ for each i, and so by n applications of the C rule, we can show that $\Gamma \longrightarrow \Delta$ is a theorem of S(M+AΘ+C).

Unfortunately, the C rules cannot be entirely eliminated from proofs in S(M+AΘ+C), which lacks the Subformula Property. This limits the theoretical utility of the calculus greatly.

Derived Logical Rules	
&L	$\varphi, \psi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad (\varphi \,\&\, \psi), \Gamma \longrightarrow \Delta.$
&R	$\Gamma \longrightarrow \Delta, \varphi;\ \Gamma \longrightarrow \Delta, \psi \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, (\varphi \,\&\, \psi)$
⊃L	$\Gamma \longrightarrow \Delta, \varphi;\ \psi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad (\varphi \supset \psi), \Gamma \longrightarrow \Delta$
⊃R	$\varphi, \Gamma \longrightarrow \Delta, \psi \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, (\varphi \supset \psi).$
≡L	$\varphi, \psi, \Gamma \longrightarrow \Delta;\ \Gamma \longrightarrow \Delta, \varphi, \psi \quad \Rightarrow \quad (\varphi \equiv \psi), \Gamma \longrightarrow \Delta$
≡R	$\varphi, \Gamma \longrightarrow \Delta, \psi;\ \psi, \Gamma \longrightarrow \Delta, \varphi \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, (\varphi \equiv \psi)$
∀L	$\varphi[^x_u], \Gamma \longrightarrow \Delta \quad \Rightarrow \quad \forall x\, \varphi, \Gamma \longrightarrow \Delta$
∀R	$\Gamma \longrightarrow \Delta, \varphi[^x_u] \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, \forall x\, \varphi$ if u does not occur in φ, Γ, Δ.

Figure 2: Derived rules of SM

Before addressing this problem, let us just note that with standard abbreviations, the rules listed in Figure 2 can all be derived in SM

4 Regaining the Subformula Property

The barrier to the elimination of C from S(M+AΘ+C), are cuts involving the formulas of Θ. For example, the following cut cannot be eliminated. Let π be the proof

$$\dfrac{\dfrac{}{u{:}\, \exists x\, \Diamond_r x, \exists x\, u{:}\, \Diamond_r x, v{:}\, \Diamond_r u, u{:}\, \Diamond_r v \longrightarrow u{:}\, \exists x\, \Diamond_r x}\,\text{I} \qquad \dfrac{\dfrac{}{v{:}\, \Diamond_r u, u{:}\, \Diamond_r v \longrightarrow u{:}\, \exists x\, \Diamond_r x, u{:}\, \exists x\, \Diamond_r x, u{:}\, \Diamond_r v}\,\text{I}}{v{:}\, \Diamond_r u, u{:}\, \Diamond_r v \longrightarrow u{:}\, \exists x\, \Diamond_r x, u{:}\, \exists x\, \Diamond_r x, \exists x\, u{:}\, \Diamond_r x}\,\exists\text{R}}{\dfrac{(u{:}\, \exists x\, \Diamond_r x \equiv \exists x\, u{:}\, \Diamond_r x), v{:}\, \Diamond_r u, u{:}\, \Diamond_r v \longrightarrow u{:}\, \exists x\, \Diamond_r x}{\forall z\,(z{:}\, \exists x\, \Diamond_r x \equiv \exists x\, z{:}\, \Diamond_r x), v{:}\, \Diamond_r u, u{:}\, \Diamond_r v \longrightarrow u{:}\, \exists x\, \Diamond_r x}\,\forall\text{L}}\,\equiv\text{L}$$

then

$$\dfrac{\dfrac{}{\longrightarrow \forall z\,(z{:}\, \exists x\, \Diamond_r x \equiv \exists x\, z{:}\, \Diamond_r x)}\,\Theta \qquad \dfrac{\dfrac{\dfrac{\dfrac{\pi}{\forall z\,(z{:}\, \exists x\, \Diamond_r x \equiv \exists x\, z{:}\, \Diamond_r x), (v{:}\, \Diamond_r u \,\&\, u{:}\, \Diamond_r v) \longrightarrow u{:}\, \exists x\, \Diamond_r x}\,\&\text{L}}{\forall z\,(z{:}\, \exists x\, \Diamond_r x \equiv \exists x\, z{:}\, \Diamond_r x), \exists y\,(y{:}\, \Diamond_r u \,\&\, u{:}\, \Diamond_r y) \longrightarrow u{:}\, \exists x\, \Diamond_r x}\,\exists\text{L}}{\forall z\,(z{:}\, \exists x\, \Diamond_r x \equiv \exists x\, z{:}\, \Diamond_r x), \forall x\, \exists y\,(y{:}\, \Diamond_r x \,\&\, x{:}\, \Diamond_r y) \longrightarrow u{:}\, \exists x\, \Diamond_r x}\,\forall\text{L}}{\forall z\,(z{:}\, \exists x\, \Diamond_r x \equiv \exists x\, z{:}\, \Diamond_r x), \forall x\, \exists y\,(y{:}\, \Diamond_r x \,\&\, x{:}\, \Diamond_r y) \longrightarrow \forall y\, y{:}\, \exists x\, \Diamond_r x}\,\forall\text{R}}{\forall x\, \exists y\,(y{:}\, \Diamond_r x \,\&\, x{:}\, \Diamond_r y) \longrightarrow \forall y\, y{:}\, \exists x\, \Diamond_r x}\,\text{C}$$

[8]Theorem 2 is needed for logical closure. For example, we need to know that if $(\varphi \supset \psi), \varphi \longrightarrow \psi$ is an axiom (MP) and $\longrightarrow \varphi$ and $\longrightarrow (\varphi \supset \psi)$ are theorems then so is $\longrightarrow \psi$.

The cut can be pushed up the tree as far as the application of ∀L, but no further:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\longrightarrow \forall z\,(z\colon \exists x\,\Diamond_r x \equiv \exists x\, z\colon \Diamond_r x)}\,\Theta \qquad \pi}{v\colon \Diamond_r u,\, u\colon \Diamond_r v \longrightarrow u\colon \exists x\,\Diamond_r x}\,\mathrm{C}}{(v\colon \Diamond_r u \mathbin{\&} u\colon \Diamond_r v) \longrightarrow u\colon \exists x\,\Diamond_r x}\,\&\mathrm{L}}{\exists y\,(y\colon \Diamond_r u \mathbin{\&} u\colon \Diamond_r y) \longrightarrow u\colon \exists x\,\Diamond_r x}\,\exists\mathrm{L}}{\forall x\,\exists y\,(y\colon \Diamond_r x \mathbin{\&} x\colon \Diamond_r y) \longrightarrow u\colon \exists x\,\Diamond_r x}\,\forall\mathrm{L}}{\forall x\,\exists y\,(y\colon \Diamond_r x \mathbin{\&} x\colon \Diamond_r y) \longrightarrow \forall y\, y\colon \exists x\,\Diamond_r x}\,\forall\mathrm{R}
$$

More generally, we have the following result:

Lemma 7 Any proof in S(M+AΘ+C) can be transformed into a proof in which all cuts have the form

$$
\dfrac{\dfrac{}{\Gamma \longrightarrow \Delta, \varphi}\,\Theta \qquad \overset{\pi}{\varphi, \Gamma' \longrightarrow \Delta'}}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\mathrm{C}
$$

where φ is a formula of Θ and is the principal formula of the final rule of π.

Proof This is a standard result for adding axioms to a system of rules from which C is eliminated ([15]). The 'axiom cuts', as they are called, are the only cuts whose rank cannot be reduced by the standard elimination algorithm.

To go further, we need to replace the axioms from Θ with rules that do the same work. In the above example, we need a rule for proving sequents of the form $\Gamma \longrightarrow \Delta, u\colon \exists x\,\varphi$. Working backwards, we can find out what is required:

$$
\dfrac{\dfrac{}{\longrightarrow \forall z\,(z\colon \exists x\,\varphi \equiv \exists x\, z\colon \varphi)}\,\Theta \qquad \dfrac{\dfrac{\dfrac{}{u\colon \exists x\,\varphi, \exists x\, u\colon \varphi, \Gamma \longrightarrow \Delta, u\colon \exists x\,\varphi}\,\mathrm{I} \quad \dfrac{\Gamma \longrightarrow \Delta, u\colon \exists x\,\varphi, u\colon \exists x\,\varphi, u\colon \varphi[^x_v]}{\Gamma \longrightarrow \Delta, u\colon \exists x\,\varphi, u\colon \exists x\,\varphi, \exists x\, u\colon \varphi}\,\exists\mathrm{R}}{(u\colon \exists x\,\varphi \equiv \exists x\, u\colon \varphi), \Gamma \longrightarrow \Delta, u\colon \exists x\,\varphi}\,\equiv\mathrm{L}}{\forall z\,(z\colon \exists x\,\varphi \equiv \exists x\, z\colon \varphi), \Gamma \longrightarrow \Delta, u\colon \exists x\,\varphi}\,\forall\mathrm{L}}{\Gamma \longrightarrow \Delta, u\colon \exists x\,\varphi}\,\mathrm{C}
$$

The derivation suggests the rule

$$
\Gamma \longrightarrow \Delta, u\colon \exists x\,\varphi, u\colon \exists x\,\varphi, u\colon \varphi[^x_v] \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, u\colon \exists x\,\varphi
$$

But we can do some cleaning up. The rule is derivable from the simpler rule

$$
\Gamma \longrightarrow \Delta, u\colon \varphi[^x_v] \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, u\colon \exists x\,\varphi
$$

which we call $:\exists$R. Given this rule, our example proof can be rewritten without the cut, as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{v\colon\Diamond_r u, u\colon\Diamond_r v \longrightarrow u\colon\Diamond_r v}\;\mathrm{I}}{v\colon\Diamond_r u, u\colon\Diamond_r v \longrightarrow u\colon\exists x\,\Diamond_r x}\;:\exists\mathrm{R}}{(v\colon\Diamond_r u\,\&\,u\colon\Diamond_r v) \longrightarrow u\colon\exists x\,\Diamond_r x}\;\&\mathrm{L}}{\exists y\,(y\colon\Diamond_r u\,\&\,u\colon\Diamond_r y) \longrightarrow u\colon\exists x\,\Diamond_r x}\;\exists\mathrm{L}}{\forall x\,\exists y\,(y\colon\Diamond_r x\,\&\,x\colon\Diamond_r y) \longrightarrow u\colon\exists x\,\Diamond_r x}\;\forall\mathrm{L}}{\forall x\,\exists y\,(y\colon\Diamond_r x\,\&\,x\colon\Diamond_r y) \longrightarrow \forall y\,y\colon\exists x\,\Diamond_r x}\;\forall\mathrm{R}$$

Using the same method, we get a rule for $:\exists$ on the left:

$$:\exists\mathrm{L} \quad u\colon\varphi[^x_v], \Gamma \longrightarrow \Delta \quad\Rightarrow\quad u\colon\exists x\,\varphi, \Gamma \longrightarrow \Delta$$

with the restriction that v does not occur in φ, Γ, Δ. Together, these two rules will replace the need for the axiom from Θ, with the advantage that they obey a subformula property. (This will be proved later, in Theorem 10.)

We can find similar rules for all the hybrid operators. They are listed in Figure 3. Some of the rules have been split into a Hybrid Rule and an Interface Rule. For example, an analysis of proofs of $\Diamond_r\varphi$ on the left gives us the rule

$$u\colon\Diamond_r\varphi, ruv, v\colon\varphi, \Gamma \longrightarrow \Delta \quad\Rightarrow\quad u\colon\Diamond_r\varphi, \Gamma \longrightarrow \Delta$$

We split this into the two rules

$$\begin{array}{llll} :\Diamond_r\mathrm{L} & u\colon\Diamond_r v, v\colon\varphi, \Gamma \longrightarrow \Delta \quad\Rightarrow\quad u\colon\Diamond_r\varphi, \Gamma \longrightarrow \Delta & \text{Hybrid} \\ A_2\mathrm{L} & ruv, \Gamma \longrightarrow \Delta \quad\Rightarrow\quad u\colon\Diamond_r v, \Gamma \longrightarrow \Delta & \text{Interface} \end{array}$$

from which the general rule can be derived. The advantage of this split is that the Hybrid Rule uses only formulas of the form $u\colon\varphi$. Formulas such as ruv that are not of this form are restricted to applications of the Interface Rules. This gives us a useful tightening of the Subformula Property that we will make good use of shortly.

Let $:H$ be the set of Hybrid Rules, and let $:H/M$ be the set of interface rules. The above discussion motivates the following series of results:

Lemma 8 A sequent of M is a theorem of S(M+AΘ+C) if and only if it is a theorem of S(M+:H+:H/M+C).

Proof We need only show that each of the rules in $:H$ and $:H/M$ is derivable in S(M+AΘ+C), and that each of the axioms of AΘ is derivable in S(M+:H+:H/M+C). We have already given an example of a derivation of the first kind; our method of discovering the rules is to construct such a derivation. Here is an example of the derivation of an axiom from a rule. Let π be

$$\dfrac{\dfrac{\dfrac{\dfrac{}{ruv, v\colon\varphi, \Gamma \longrightarrow \Delta, ruv}\;\mathrm{I}}{u\colon\Diamond_r v, v\colon\varphi, \Gamma \longrightarrow \Delta, ruv}\;A_2\mathrm{L} \qquad \dfrac{}{u\colon\Diamond_r v, v\colon\varphi, \Gamma \longrightarrow \Delta, v\colon\varphi}\;\mathrm{I}}{u\colon\Diamond_r v, v\colon\varphi, \Gamma \longrightarrow \Delta, (ruv\,\&\,v\colon\varphi)}\;\&\mathrm{R}}{u\colon\Diamond_r v, v\colon\varphi, \Gamma \longrightarrow \Delta, \exists x\,(rux\,\&\,x\colon\varphi)}\;\exists\mathrm{R}$$

Hybrid Logical Rules $:H$	
$:\mathrm{L}_1$	$u\colon v, \Gamma[^w_u] \longrightarrow \Delta[^w_u] \quad \Rightarrow \quad u\colon v, \Gamma[^w_v] \longrightarrow \Delta[^w_v]$.
$:\mathrm{L}_2$	$u\colon v, \Gamma[^w_v] \longrightarrow \Delta[^w_v] \quad \Rightarrow \quad u\colon v, \Gamma[^w_u] \longrightarrow \Delta[^w_u]$.
$:\mathrm{R}$	$\Rightarrow \quad \Gamma \longrightarrow \Delta, u\colon u$
$:\sim\mathrm{L}$	$\Gamma \longrightarrow \Delta, u\colon \varphi \quad \Rightarrow \quad u\colon \sim\varphi, \Gamma \longrightarrow \Delta$
$:\sim\mathrm{R}$	$u\colon \varphi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, u\colon \sim\varphi$
$:\vee\mathrm{L}$	$u\colon \varphi, \Gamma \longrightarrow \Delta;\; u\colon \psi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad u\colon (\varphi \vee \psi), \Gamma \longrightarrow \Delta$
$:\vee\mathrm{R}$	$\Gamma \longrightarrow \Delta, u\colon \varphi, u\colon \psi \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, u\colon (\varphi \vee \psi)$
$:\exists\mathrm{L}$	$u\colon \varphi[^x_v], \Gamma \longrightarrow \Delta \quad \Rightarrow \quad u\colon \exists x\, \varphi, \Gamma \longrightarrow \Delta$ if v does not occur in φ, Γ, Δ.
$:\exists\mathrm{R}$	$\Gamma \longrightarrow \Delta, u\colon \varphi[^x_v] \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, u\colon \exists x\, \varphi$
$:\Diamond\mathrm{L}$	$v\colon \varphi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad u\colon \Diamond\varphi, \Gamma \longrightarrow \Delta$ if v does not occur in φ, Γ, Δ.
$:\Diamond\mathrm{R}$	$\Gamma \longrightarrow \Delta, v\colon \varphi \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, u\colon \Diamond\varphi$
$:\Diamond_r\mathrm{L}$	$u\colon \Diamond_r v, v\colon \varphi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad u\colon \Diamond_r\varphi, \Gamma \longrightarrow \Delta$ if v does not occur in φ, Γ, Δ.
$:\Diamond_r\mathrm{R}$	$\Gamma \longrightarrow \Delta, v\colon \varphi;\; \Gamma \longrightarrow \Delta, u\colon \Diamond_r v \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, u\colon \Diamond_r\varphi$
$:@\mathrm{L}$	$v\colon \varphi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad u\colon @_v\varphi, \Gamma \longrightarrow \Delta$
$:@\mathrm{R}$	$\Gamma \longrightarrow \Delta, v\colon \varphi \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, u\colon @_v\varphi$
$:\downarrow\mathrm{L}$	$u\colon \varphi[^x_u], \Gamma \longrightarrow \Delta \quad \Rightarrow \quad u\colon \downarrow_x \varphi, \Gamma \longrightarrow \Delta$
$:\downarrow\mathrm{R}$	$\Gamma \longrightarrow \Delta, u\colon \varphi[^x_u] \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, u\colon \downarrow_x \varphi$
Interface Rules $:H/M$	
$A_1\mathrm{L}$	$pu, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad u\colon p, \Gamma \longrightarrow \Delta$
$A_1\mathrm{R}$	$\Gamma \longrightarrow \Delta, pu \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, u\colon p$
$A_2\mathrm{L}$	$ruv, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad u\colon \Diamond_r v, \Gamma \longrightarrow \Delta$
$A_2\mathrm{R}$	$\Gamma \longrightarrow \Delta, ruv \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, u\colon \Diamond_r v$

Figure 3: The Hybrid Logical Rules and Interface Rules

Then

$$\dfrac{\dfrac{\dfrac{\pi}{u\colon \Diamond_r\varphi, \Gamma \longrightarrow \Delta, \exists x\,(rux\ \&\ x\colon \varphi)}\,\Diamond_r\mathrm{L} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{ruv, v\colon \varphi, \Gamma \longrightarrow \Delta, ruv}\,\mathrm{I}}{ruv, v\colon \varphi, \Gamma \longrightarrow \Delta, u\colon \Diamond_r v}\,A_2\mathrm{R} \quad \dfrac{}{ruv, v\colon \varphi, \Gamma \longrightarrow \Delta, v\colon \varphi}\,\mathrm{I}}{ruv, v\colon \varphi, \Gamma \longrightarrow \Delta, u\colon \Diamond_r\varphi}\,\Diamond_r\mathrm{R}}{(ruv\ \&\ v\colon \varphi), \Gamma \longrightarrow \Delta, u\colon \Diamond_r\varphi}\,\&\mathrm{L}}{\exists x\,(rux\ \&\ x\colon \varphi), \Gamma \longrightarrow \Delta, u\colon \Diamond_r\varphi}\,\exists\mathrm{L}}{\Gamma \longrightarrow \Delta, (u\colon \Diamond_r\varphi \equiv \exists x\,(rux\ \&\ x\colon \varphi))}\,\equiv\mathrm{R}}{\Gamma \longrightarrow \Delta, \forall z\,(z\colon \Diamond_r\varphi \equiv \exists x\,(rzx\ \&\ x\colon \varphi))}\,\forall\mathrm{R}$$

The other derivations are all similarly straightforward.

Lemma 9 C is admissible in S(M+:H+:H/M).

Proof To eliminate cuts from proofs in S(M+:H+:H/M+C) we follow the method described in the proof of Theorem 2. We have to show that (1) cuts can be pushed through the new rules when the cut formula is not principal, and (2) that cut rank can be decreased when the cut formula is the principal formula of applications of the new rules. Most of the new rules do not alter the non-principal formulas, and so (1) is straightforward—we omit the details. The only problematic case is that of the Hybrid rules $:\text{L}_1$ and $:\text{L}_2$, but they are treated in the same way as we treated the $=$L rules in the proof of Theorem 2.

Again, for most of the rules the transformations required for (2) are exactly analogous to those of their cousins in M. For example, $:\vee$L and $:\vee$R are transformed in the same way as $\vee$L and $\vee$R. The transformations required for the Interface Rules are straightforward:

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \longrightarrow \Delta, pu}}{\Gamma \longrightarrow \Delta, u{:}\,p}\,A_1\text{R} \quad \dfrac{\overset{\pi_2}{pu, \Gamma' \longrightarrow \Delta'}}{u{:}\,p, \Gamma' \longrightarrow \Delta'}\,A_1\text{L}}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\text{C} \quad \rightsquigarrow \quad \dfrac{\overset{\pi_1}{\Gamma \longrightarrow \Delta, pu} \quad \overset{\pi_2}{pu, \Gamma' \longrightarrow \Delta'}}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\text{C}$$

The A_2 rules are dealt with similarly. The only rules that deserve comment are those for the modal ($\Diamond$, $\Diamond_r$) and hybrid (@, $\downarrow$) operators, so we finish the proof with these. First the modals:

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \longrightarrow \Delta, v{:}\,\varphi}}{\Gamma \longrightarrow \Delta, u{:}\,\Diamond\varphi}\,\Diamond\text{R} \quad \dfrac{\overset{\pi_2}{w{:}\,\varphi, \Gamma' \longrightarrow \Delta'}}{u{:}\,\Diamond\varphi, \Gamma' \longrightarrow \Delta'}\,\Diamond\text{L}}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\text{C} \quad \rightsquigarrow \quad \dfrac{\overset{\pi_1}{\Gamma \longrightarrow \Delta, v{:}\,\varphi} \quad \overset{\pi_2^*[{}^w_v]}{v{:}\,\varphi, \Gamma' \longrightarrow \Delta'}}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\text{C}$$

(As before, π_i^* is the result of renaming the parameters in π_i that are introduced by a restricted rule, such as $\exists$L. Note that $\Gamma'[{}^w_v] = \Gamma'$ and $\Delta'[{}^w_v] = \Delta'$ because w cannot occur in Γ', Δ'.)

$$\dfrac{\dfrac{\overset{\pi_1'}{\Gamma \longrightarrow \Delta, u{:}\,\Diamond_r v} \quad \overset{\pi_1''}{\Gamma \longrightarrow \Delta, v{:}\,\varphi}}{\Gamma \longrightarrow \Delta, u{:}\,\Diamond_r\varphi}\,\Diamond_r\text{R} \quad \dfrac{\overset{\pi_2}{u{:}\,\Diamond_r w, w{:}\,\varphi, \Gamma' \longrightarrow \Delta'}}{u{:}\,\Diamond_r\varphi, \Gamma' \longrightarrow \Delta'}\,\Diamond_r\text{L}}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\text{C}$$

$$\rightsquigarrow \quad \dfrac{\overset{\pi_1'}{\Gamma \longrightarrow \Delta, u{:}\,\Diamond_r v} \quad \dfrac{\overset{\pi_1''}{\Gamma \longrightarrow \Delta, v{:}\,\varphi} \quad \overset{\pi_2^*[{}^w_v]}{u{:}\,\Diamond_r v, v{:}\,\varphi, \Gamma' \longrightarrow \Delta'}}{u{:}\,\Diamond_r v, \Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\text{C}}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\text{C}$$

Now for the hybrid operators:

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \longrightarrow \Delta, v{:}\,\varphi}}{\Gamma \longrightarrow \Delta, u{:}\,@_v\varphi}\,@\text{R} \qquad \dfrac{\overset{\pi_2}{v{:}\,\varphi, \Gamma' \longrightarrow \Delta'}}{u{:}\,@_v\varphi, \Gamma'\Delta' \longrightarrow}\,@\text{L}}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\text{C} \quad \rightsquigarrow \quad \dfrac{\overset{\pi_1}{\Gamma \longrightarrow \Delta, v{:}\,\varphi} \qquad \overset{\pi_2}{v{:}\,\varphi, \Gamma' \longrightarrow \Delta'}}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\text{C}$$

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \longrightarrow \Delta, u{:}\,\varphi}}{\Gamma \longrightarrow \Delta, u{:}\,\downarrow_x \varphi}\downarrow\text{R} \qquad \dfrac{\overset{\pi_2}{u{:}\,\varphi, \Gamma' \longrightarrow \Delta'}}{u{:}\,\downarrow_x \varphi, \Gamma'\Delta' \longrightarrow}\downarrow\text{L}}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\text{C} \quad \rightsquigarrow \quad \dfrac{\overset{\pi_1}{\Gamma \longrightarrow \Delta, u{:}\,\varphi} \qquad \overset{\pi_2}{u{:}\,\varphi, \Gamma' \longrightarrow \Delta'}}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,\text{C}$$

Theorem 10 A sequent of M is Θ-valid if and only if it is a theorem of S(M+:H + :H/M).

Proof Soundness (the 'if' direction) follows from the fact that each of the rules in :H and :H/M is derivable in S(M+AΘ+C), which is sound by Theorem 6. For completeness, suppose that $\Gamma \longrightarrow \Delta$ is Θ-valid. By Theorem 6, there is a proof of this sequent in S(M+AΘ+C), which can be converted into a proof in S(M+:H+:H/M+C) by Lemma 8, and then into a proof without C, by Lemma 9.

The calculus S(M+:H+:H/M) gives a complete characterization of the valid sequents of M and comes very close to having the Subformula Property. The rules of SM have the Subformula Property, as do those of :H, with a suitable definition of the subformulas of $u{:}\,\varphi$. We say that $v{:}\,\psi$ is a *subformula* of $u{:}\,\varphi$ if ψ is the result of replacing zero or more occurrences of the parameters of a (genuine) subformula of φ. The Interface Rules lack the Subformula Property, but this is not too serious; we can replace them with axioms that have it (see Figure 4).

Interface Axioms A:H/M		
A_1^*L	$\Rightarrow$	$u{:}\,p, \Gamma \longrightarrow \Delta, pu$
A_1^*R	$\Rightarrow$	$pu, \Gamma \longrightarrow \Delta, u{:}\,p$
A_2^*L	$\Rightarrow$	$u{:}\,\Diamond_r v, \Gamma \longrightarrow \Delta, ruv$
A_2^*R	$\Rightarrow$	$ruv, \Gamma \longrightarrow \Delta, u{:}\,\Diamond_r v$

Figure 4: Interface Axioms A:H/M

Corollary 11 A sequent of M is Θ-valid if and only if it is a theorem of S(M+:H + A:H/M).

Proof From Theorem 10. We have to show that applications of the Interface Rules can be replaced by applications of the Interface Axioms. This is simply a matter of showing that the Interface Rules commute with all the other rules until they get to the leaves of the proof tree. All leaves end in axioms: I,=R,: R, and the Interface

Axioms. In each case, the Interface Rule can be eliminated using another axiom. For example,

$$\dfrac{\overline{pu, \Gamma \longrightarrow \Delta, pu}\ \mathrm{I}}{u\colon p, \Gamma \longrightarrow \Delta, pu}\ A_1\mathrm{L} \quad \rightsquigarrow \quad \overline{u\colon p, \Gamma \longrightarrow \Delta, pu}\ A_1^*\mathrm{L}$$

$$\dfrac{\overline{u\colon \Diamond_r v, \Gamma \longrightarrow \Delta, ruv}\ \mathrm{A}_2^*\mathrm{L}}{u\colon \Diamond_r v, \Gamma \longrightarrow \Delta, u\colon \Diamond_r v}\ A_2\mathrm{R} \quad \rightsquigarrow \quad \overline{u\colon \Diamond_r v, \Gamma \longrightarrow \Delta, u\colon \Diamond_r v}\ \mathrm{I}$$

The system S(M+:H+A:H/M) has the Subformula Property and so we have reached the goal of this section.

5 Internalization and Rules For All

Now that Θ-validity in M has been captured with a respectable sequent calculus, we can use Lemma 1 to provide a calculus for validity in H. Let S:H be the set consisting of the Structural Rules, I and S, together with the Hybrid Rules :H.

Theorem 12 Suppose that u does not occur in Γ, Δ. The sequent $\Gamma \longrightarrow \Delta$ of H is valid if and only if $u\colon \Gamma \longrightarrow u\colon \Delta$ is a theorem of S:H. Moreover, the proof uses only the rules for the operators occurring in Γ, Δ.

Proof By Corollary 11, the calculus S(M+S:H+A:H/M) is sufficient, but this calculus enjoys the Subformula Property, and so any proof of a sequent of the form $u\colon \Gamma \longrightarrow u\colon \Delta$ uses only the rules of S:H.

This is a great improvement on S(M+AΘ+C), but still slightly unsatisfactory because the proofs of S:H are not fully internalized. The formulas occurring in the proof are not formulas of H; they are all of the form $u\colon \varphi$, which belongs to the formal metalanguage M.

To push internalization as far as possible, we will reformulate the rules of S:H using the hybrid operator @. This gives us the system S@H, shown in Figure 5. Without @ we would have to look much more closely at the structure of proofs to make any further progress toward internalization.

S@H is fully internalized: the only formulas occurring in proofs are formulas of H. What's more, it has the Subformula Property, if we reinterpret the definition of subformula appropriately. The only remaining drawback is that the calculus only applies to a fragment of the language: the formulas of the form $@_u\varphi$. This is sufficient for the purpose of characterizing validity, because every sequent $\Gamma \longrightarrow \Delta$ is equivalent to a sequent in this fragment, namely, $@_u\Gamma \longrightarrow @_u\Delta$. Yet the absence of rules for dealing with sequents without @ is regrettable, and unnecessary. In the final tuning of our proof-theoretic apparatus, we aim for a more egalitarian logic in which there are 'Rules for All'.

Nominals—individual variables occurring as formulas—play an essential part in liberating the calculus from @-prefixed formulas. A single nominal parameter u on the left side of a sequent is enough to anchor all non-@-prefixed formulas to the same element and so removes"' the need for them to share a prefix. To shift between @-prefixed formulas and free nominals we need the Nominal Rules N, shown in Figure 6.

Internalized Hybrid Logical Rules $@H$	
$@\text{L}_1$	$@_u v, \Gamma[^w_u] \longrightarrow \Delta[^w_u] \quad \Rightarrow \quad @_u v, \Gamma[^w_v] \longrightarrow \Delta[^w_v]$.
$@\text{L}_2$	$@_u v, \Gamma[^w_v] \longrightarrow \Delta[^w_v] \quad \Rightarrow \quad @_u v, \Gamma[^w_u] \longrightarrow \Delta[^w_u]$.
$@\text{R}$	$\Rightarrow \quad \Gamma \longrightarrow \Delta, @_u u$
$@{\sim}\text{L}$	$\Gamma \longrightarrow \Delta, @_u\varphi \quad \Rightarrow \quad @_u{\sim}\varphi, \Gamma \longrightarrow \Delta$
$@{\sim}\text{R}$	$@_u\varphi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, @_u{\sim}\varphi$
$@\vee\text{L}$	$@_u\varphi, \Gamma \longrightarrow \Delta;\ @_u\psi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad @_u(\varphi \vee \psi), \Gamma \longrightarrow \Delta$
$@\vee\text{R}$	$\Gamma \longrightarrow \Delta, @_u\varphi, @_u\psi \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, @_u(\varphi \vee \psi)$
$@\exists\text{L}$	$@_u\varphi[^x_v], \Gamma \longrightarrow \Delta \quad \Rightarrow \quad @_u\exists x\,\varphi, \Gamma \longrightarrow \Delta$ if v does not occur in φ, Γ, Δ.
$@\exists\text{R}$	$\Gamma \longrightarrow \Delta, @_u\varphi[^x_v] \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, @_u\exists x\,\varphi$
$@\Diamond\text{L}$	$@_v\varphi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad @_u\Diamond\varphi, \Gamma \longrightarrow \Delta$ if v does not occur in φ, Γ, Δ.
$@\Diamond\text{R}$	$\Gamma \longrightarrow \Delta, @_v\varphi \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, @_u\Diamond\varphi$
$@\Diamond_r\text{L}$	$@_u\Diamond_r v, @_v\varphi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad @_u\Diamond_r\varphi, \Gamma \longrightarrow \Delta$ if v does not occur in φ, Γ, Δ.
$@\Diamond_r\text{R}$	$\Gamma \longrightarrow \Delta, @_v\varphi;\ \Gamma \longrightarrow \Delta, @_u\Diamond_r v \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, @_u\Diamond_r\varphi$
$@@\text{L}$	$@_v\varphi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad @_u@_v\varphi, \Gamma \longrightarrow \Delta$
$@@\text{R}$	$\Gamma \longrightarrow \Delta, @_v\varphi \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, @_u@_v\varphi$
$@\downarrow\text{L}$	$@_u\varphi[^x_u], \Gamma \longrightarrow \Delta \quad \Rightarrow \quad @_u \downarrow_x \varphi, \Gamma \longrightarrow \Delta$
$@\downarrow\text{R}$	$\Gamma \longrightarrow \Delta, @_u\varphi[^x_u] \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, @_u \downarrow_x \varphi$

Figure 5: Internalized Hybrid Logical Rules $@H$

Lemma 13 A sequent of H is valid if and only if it is a theorem of S($@H$+N).

Proof First note that a sequent $\Gamma \longrightarrow \Delta$ of H is valid if and only if $@_u\Gamma \longrightarrow @_u\Delta$ is a theorem of S$@H$ for u not in Γ, Δ. This follows from Theorem 12 together with the observation that the rules, and hence the proofs, in these two systems are isomorphic. We can convert one to the other simply by replacing u: by $@_u$ and vice versa. But now suppose that $\Gamma \longrightarrow \Delta$ is valid and so $@_u\Gamma \longrightarrow @_u\Delta$ is a theorem of S$@H$ (for u not in Γ, Δ). Weakening, we get $u, @_u\Gamma \longrightarrow @_u\Delta$, from which we can derive $u, \Gamma \longrightarrow \Delta$ using multiple applications of ^@L and ^@R. Finally, we get $\Gamma \longrightarrow \Delta$ using name. For the converse, we need only check that each rule of N is sound.

Now, with the Nominal Rules in place, the rules of $@H$ can be converted to use the nominal-based method of context sharing. We convert each rule of $@H$ of the form

$$@_u\Gamma_1, \Gamma \longrightarrow \Delta, @_u\Delta_1 \quad \Rightarrow \quad @_u\Gamma_2, \Gamma' \longrightarrow \Delta', @_u\Delta_2$$

to the rule

$$u, \Gamma_1, \Gamma \longrightarrow \Delta, \Delta_1 \quad \Rightarrow \quad u, \Gamma_2, \Gamma' \longrightarrow \Delta', \Delta_2$$

If u does not occur in (the general statement of) this rule except in the place we have shown, we can go one step further and convert the rule to

$$\Gamma_1, \Gamma \longrightarrow \Delta, \Delta_1 \quad \Rightarrow \quad \Gamma_2, \Gamma' \longrightarrow \Delta', \Delta_2$$

Nominal Rules N	
$^\vee$@L	$u, \varphi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad u, @_u\varphi, \Gamma \longrightarrow \Delta$
$^\vee$@R	$u, \Gamma \longrightarrow \Delta, \varphi \quad \Rightarrow \quad u, \Gamma \longrightarrow \Delta, @_u\varphi$
$^\wedge$@L	$u, @_u\varphi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad u, \varphi, \Gamma \longrightarrow \Delta$
$^\wedge$@R	$u, \Gamma \longrightarrow \Delta, @_u\varphi \quad \Rightarrow \quad u, \Gamma \longrightarrow \Delta, \varphi$
name	$u, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad \Gamma \longrightarrow \Delta$ if u does not occur in Γ, Δ.
term	$u, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad \Gamma \longrightarrow \Delta$ if all formulas in Γ, Δ are @-prefixed.

Figure 6: Nominal Rules N

(Here the shared element of the non-@-prefixed formulas is given implicitly as the point of evaluation.) This gives us the Nominal-based Internalized Hybrid Logic Rules NH, shown in Figure 7.

Nominal-based Internalized Hybrid Logic Rules NH	
NL	$u, v, \Gamma[^w_u] \longrightarrow \Delta[^w_u] \quad \Rightarrow \quad u, v, \Gamma[^w_v] \longrightarrow \Delta[^w_v]$.
$\sim$L	$\Gamma \longrightarrow \Delta, \varphi \quad \Rightarrow \quad \sim\varphi, \Gamma \longrightarrow \Delta$
$\sim$R	$\varphi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, \sim\varphi$
$\vee$L	$\varphi, \Gamma \longrightarrow \Delta;\ \psi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad (\varphi \vee \psi), \Gamma \longrightarrow \Delta$
$\vee$R	$\Gamma \longrightarrow \Delta, \varphi, \psi \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, (\varphi \vee \psi)$
$\exists$L	$\varphi[^x_v], \Gamma \longrightarrow \Delta \quad \Rightarrow \quad \exists x\, \varphi, \Gamma \longrightarrow \Delta$ if v does not occur in φ, Γ, Δ.
$\exists$R	$\Gamma \longrightarrow \Delta, \varphi[^x_v] \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, \exists x\, \varphi$
$\Diamond$L	$@_v\varphi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad \Diamond\varphi, \Gamma \longrightarrow \Delta$ if v does not occur in φ, Γ, Δ.
$\Diamond$R	$\Gamma \longrightarrow \Delta, @_v\varphi \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, \Diamond\varphi$
$\Diamond_r$L	$\Diamond_r v, @_v\varphi, \Gamma \longrightarrow \Delta \quad \Rightarrow \quad \Diamond_r\varphi, \Gamma \longrightarrow \Delta$ if v does not occur in φ, Γ, Δ.
$\Diamond_r$R	$\Gamma \longrightarrow \Delta, @_v\varphi;\ \Gamma \longrightarrow \Delta, \Diamond_r v \quad \Rightarrow \quad \Gamma \longrightarrow \Delta, \Diamond_r\varphi$
$N{\downarrow}$L	$u, \varphi[^x_u], \Gamma \longrightarrow \Delta \quad \Rightarrow \quad u, \downarrow_x \varphi, \Gamma \longrightarrow \Delta$
$N{\downarrow}$R	$u, \Gamma \longrightarrow \Delta, \varphi[^x_u] \quad \Rightarrow \quad u, \Gamma \longrightarrow \Delta, \downarrow_x \varphi$

Figure 7: Nominal-based Internalized Hybrid Logic Rules NH

After conversion several of the rules become redundant. The result of converting @R is just a special case of I, and the @@ rules become entirely trivial. These have been omitted from the above table.

Many of the rules in NH are very familiar. The operators of classical logic involve no hybrid interaction and so have returned to a familiar form. Interestingly, the rules for the modal operators use both nominals and the @ operator.

Theorem 14 A sequent of H is valid if and only if it is a theorem of S$(N + NH)$.

Proof We show that any proof in S$(N + NH)$ can be converted into a proof in

S($@H + N$) and vice versa. The theorem follows from Lemma 13. Each rule

$$R \quad @_u\Gamma_1, \Gamma \longrightarrow \Delta, @_u\Delta_1 \quad \Rightarrow \quad @_u\Gamma_2, \Gamma' \longrightarrow \Delta', @_u\Delta_2$$

of $@H$ is first converted into the rule

$$NR \quad u, \Gamma_1, \Gamma \longrightarrow \Delta, \Delta_1 \quad \Rightarrow \quad u, \Gamma_2, \Gamma' \longrightarrow \Delta', \Delta_2$$

Wherever NR occurs in a proof, it can be replaced by the following derivation from S($@H + N$):

$$\begin{array}{c l}
u, \Gamma_1, \Gamma \longrightarrow \Delta, \Delta_1 & \\
\hline
u, @_u\Gamma_1, @_u\Gamma \longrightarrow @_u\Delta, @_u\Delta_1 & {}^\vee@\mathrm{L}{}^\vee@\mathrm{R} \\
\hline
@_u\Gamma_1, @_u\Gamma \longrightarrow @_u\Delta, @_u\Delta_1 & \text{term} \\
\hline
@_u\Gamma_2, @_u\Gamma' \longrightarrow @_u\Delta', @_u\Delta_2 & R \\
\hline
u, @_u\Gamma_2, @_u\Gamma' \longrightarrow @_u\Delta', @_u\Delta_2 & \mathrm{W} \\
\hline
u, \Gamma_2, \Gamma' \longrightarrow \Delta', \Delta_2 & {}^\wedge@\mathrm{L}{}^\wedge@\mathrm{R}
\end{array}$$

(The step using R is okay because for all the rules in S$@H$, the prefixing of $@_u$ to the Γ, Δ and Γ', Δ' does not alter the applicability of the rule.) Similarly, wherever R occurs in a proof, it can be replaced by the following derivation from S($NH + N$), with v new to the proof:

$$\begin{array}{c l}
@_u\Gamma_1, \Gamma \longrightarrow \Delta, @_u\Delta_1 & \\
\hline
v, @_u\Gamma_1, @_v\Gamma \longrightarrow @_v\Delta, @_u\Delta_1 & \mathrm{W}, {}^\vee @\mathrm{L}, {}^\vee @\mathrm{R} \\
\hline
@_u\Gamma_1, @_v\Gamma \longrightarrow @_v\Delta, @_u\Delta_1 & \text{term} \\
\hline
u, \Gamma_1, @_v\Gamma \longrightarrow @_v\Delta, \Delta_1 & \mathrm{W}, {}^\wedge @\mathrm{L}, {}^\wedge @\mathrm{R} \\
\hline
u, \Gamma_2, @_v\Gamma \longrightarrow @_v\Delta, \Delta_2 & NR \\
\hline
u, @_u\Gamma_2, @_v\Gamma \longrightarrow @_v\Delta, @_u\Delta_2 & {}^\vee@\mathrm{L}{}^\vee@\mathrm{R} \\
\hline
@_u\Gamma_2, @_v\Gamma \longrightarrow @_v\Delta, @_u\Delta_2 & \text{term} \\
\hline
v, @_u\Gamma_2, \Gamma \longrightarrow \Delta, @_u\Delta_2 & \mathrm{W}, {}^\wedge @\mathrm{L}, {}^\wedge @\mathrm{R} \\
\hline
@_u\Gamma_2, \Gamma \longrightarrow \Delta, @_u\Delta_2 & \text{name}
\end{array}$$

(Again, the application of NR is okay because these rules are similarly immune to prefixing the Γ, Δ part.) Finally, in the case that u does not occur in Γ, Δ we need to show that $\Gamma \longrightarrow \Delta$ can be derived from $u, \Gamma \longrightarrow \Delta$ and vice versa. But this is just W in one direction and name in the other.

The price of implementing our policy of Rules for All is that the calculus no longer enjoys the Subformula Property. A proof may contain any number of @-prefixes not in the end sequent, introduced using the $^\wedge@$ rules. Of course, we know that excessive introduction of prefixes is unnecessary, and from the proofs of Lemma 13 and Theorem 14 we see that only one layer of prefixes is ever needed.

The internalization strategy, illustrated here for hybrid logics, can be applied to a wide range of logical operators. Any first-order definable operator can be tackled in this way, although it is presently unclear to me how far one can go in any given case. For a candidate operator with a first-order definition, it is easy to find rules using the technique shown on page 10. Yet this process is not entirely automated. There is still a little 'tidying up' to be done, and further work showing that Cut can be eliminated. Some problems arise with operators having nested quantifiers because of the need to keep track of dependant variables, but the limits of the method have not yet been established.

For hybrid logics, full internalization is possible only because of the expressive power of the language, specifically the fine control exercised by @ and the nominals. For @-less fragments and other modal logics, there may be no way of duplicating the function of the metalogical : within the object language. In that case, the internalization strategy will halt, awaiting a more specific analysis of the structure of proofs, from which it may be possible to guess restrictions of the kind used in the sequent calculi for S4 and intuitionistic logic. Outside the realm of normal model logics and their hybrid extensions, the boundaries are even less clear. Some preliminary success has been achieved using internalization to construct a Tableaux system for S1 based on its neighbourhood semantics (see[9]).

Acknowledgments

The present paper grew out of an earlier manuscript [14] in which the calculus presented here was shown to have a number of nice properties, and which was subsequently discussed extensively in [3]. The manuscript was hand-written during a short stay at Patrick Blackburn's house in the Lorraine, and then photocopied and distributed to a number of interested people. It has not been published or even converted to electronic form, although it is mentioned in a number of places, including [10], the Hybrid Logics' Home Page. A longer history traces the work back to [11], [12], and [13]. My thanks to Patrick for his hospitality and enthusiasm and to the referees of the paper for useful comments and corrections.

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