Axiomatizing Modal Theories of Subset Spaces (An Example of the Power of Hybrid Logic)

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Abstract

This paper is about a synthesis of two quite different modal reasoning formalisms: the logic of subset spaces, and hybrid logic. Going beyond commonly considered languages we introduce names of objects involving sets and corresponding satisfaction operators. In this way we are able to completely axiomatize the theory of certain classes of subset spaces which are difficult to deal with purely modally. We also study effectivity properties of the resulting logical systems.

1 Introduction

At a workshop on hybrid logic we need not go into the fundamental matters of this system. And we will, actually, not go beyond the hybrid material very much that is contained in Chapter 7 of the textbook [3]. Our main emphasis is to apply instead basic hybrid logic to *subset spaces;* such structures play a part in the logic of *knowledge,* and in connection with *topological reasoning* as well. Maybe these topics are less known, and the latter in particular. So, we give some comments on their relevance in the following.

The notion of *knowledge* has recently proved to be a useful concept for analyzing distributed scenarios; cf [8], and the textbooks [6] and [13]. Especially powerful tools are yielded by combining logics of knowledge and *time;* cf [9]. In fact, given a multi–agent system $\mathcal{S}$ and an agent A involved in it, the resulting formalisms allow one to describe how the actual *knowledge state* of A develops. This quantity is represented by a particular set S of states of the system $\mathcal{S}$ in the formal model. Thus, the development of S corresponds to a certain set $\mathcal{O}$ of subsets of the set X of all states of $\mathcal{S}$, where $\mathcal{O}$ is 'structured' by the underlying flow of time.

This observation links knowledge to general topology, cf [5], where the relevant domains are represented by spaces of subsets of a given set X, too, viz opens, neighbourhoods, etc. The basic task of topology is to give a mathematical framework for geometric ideas and procedures like closeness, separation, approximation of objects, etc. As spatial or temporal data bases are always structured topologically in this meaning, implicitly at least, a formal treatment of these notions, i.e., *topological reasoning,* is of fundamental relevance to any spatio–temporal reasoning system.

The modal logic of subset spaces $(X, \mathcal{O})$ of the type considered above, which was proposed in [5], can be viewed as the qualitative core of topological reasoning. In case of a single agent this system contains a connective K representing knowledge as well as a connective $\Box$ representing *effort* of the agent (to approximate an object and acquire knowledge, respectively). Both modalities interact, according to the time structure underlying the effort operator more or less explicitly.

As a modal system, the logic of subset spaces has the usual shortcomings (besides the well–known advantages). In particular, distinguished states cannot be addressed directly. Now, the question arises whether hybrid logic can rectify this deficiency similar to the case of common modal logic. Trying this we are, however, confronted with a serious problem very soon for the following reason: validity of formulas of the logic of subset spaces is defined with respect to *neighbourhood situations* x, U consisting of a state x and an 'open' set U such that $x \in U$; thus, unrestricted jumps forced by the satisfaction operator corresponding to a nominal are impossible in principle! It is true that we could introduce 'qualified' nominals, but satisfaction operators would become definable then. The addition of names of *sets* (from $\mathcal{O}$) would be a further reaching approach, which actually has not been investigated in detail yet. We will instead go another way here, naming the real semantic objects of the logic of subset spaces viz neighbourhood situations.

Why is this approach useful at all? There are several reasons. First, increasing the expressive power of the language has clearly an impact on the practical applicability of the system to concrete reasoning tasks. Second, the logic of some interesting classes of subset spaces can be axiomatized now. This is our main focus at present. E.g., no modal axiomatization is known for the class of *linear* subset spaces, by means of which sets evolving temporally can be modelled; however, hybridly we succeed. Third, proofs of completeness change in some cases and become more transparent. For example, the step–by–step method for general subset spaces, cf [5], is superseded now by canonical model techniques. [1]

What we do in the present paper is to draw up the theoretical topics just mentioned. To this end we proceed in the following way. We first introduce a hybrid language of subset spaces extending the modal one by nominals as names of neighbourhood situations and corresponding satisfaction operators. Afterwards we briefly revisit usual hybrid logic and argue that the completeness proof contained in [3] can be transferred to the present case almost liter-

[1] Patrick Blackburn pointed out to us where step–by–step is hidden in hybrid completeness à la [3], 7.3. (Personal communication.)

ally. Then, we axiomatize the hybrid logic of general subset spaces and prove its 'canonical' completeness. We finally discuss several special classes of hybrid subset spaces, in particular, linear ones. We show that in the latter case we get a decidable hybrid logic of 'low' complexity.

2 A Hybrid Language for Subset Spaces

Let PROP and NNS be two disjoint sets of symbols called *propositional variables* and *names of neighbourhood situations,* respectively. We use p, q, ... to denote typical elements of PROP, and $\mathfrak{n}, \mathfrak{m}$, ... for the same purpose in case of neighbourhood situations. Moreover, we assume that there is a distinguished name $\mathfrak{n}_0 \in$ NNS, because of technical reasons which will become clear later on. We define the set WFF of *well–formed formulas* of a hybrid bimodal language of subset spaces over PROP and NNS by the generative rule

$$\alpha \ ::= \ p \mid \mathfrak{n} \mid \neg\alpha \mid \beta \wedge \alpha \mid K\alpha \mid \Box\alpha \mid @_{\mathfrak{n}}\alpha.$$

In particular, we designate formulas by lower case Greek letters. The missing boolean connectives $\top, \bot, \vee, \rightarrow, \leftrightarrow$ are treated as abbreviations, if need be. The duals of the one–place modalities K and $\Box$ are written L and $\Diamond$, respectively.

Hybrid logic deals with *nominals* from a set NOM of names of states, and *satisfaction operators* $@_i$ where $i \in$ NOM, among other things. Presently NOM is replaced by NNS, and $@_i$ by $@_{\mathfrak{n}}$ correspondingly. There is no need to distinguish between $@_{\mathfrak{n}}$ and its dual because $@_{\mathfrak{n}}$ turns out to be self–dual, as in usual hybrid logic. It should be noted that we do not consider other hybrid binders like the $\downarrow$–operator in the technical part of this paper.

We will next give meaning to formulas. For this purpose we have to define the relevant structures first. These are essentially the same domains as of the modal logic of subset spaces, but the point–dependent only valuation of the propositional variables is to be extended to NNS suitably. Let $\mathcal{P}(A)$ designate the powerset of a given set A in the following.

Definition 2.1 [Subset frames; hybrid subset spaces]

(i) Let X be a non–empty set and $\mathcal{O} \subseteq \mathcal{P}(X)$ a system of non–empty subsets of X such that $X \in \mathcal{O}$. Then the pair $(X, \mathcal{O})$ is called a *subset frame.*

(ii) Let $(X, \mathcal{O})$ be a subset frame. The set of *neighbourhood situations* of $(X, \mathcal{O})$ is $\mathcal{N} := \{x, U \mid (x, U) \in X \times \mathcal{O} \text{ and } x \in U\}$. (Neighbourhood situations are written without brackets.)

(iii) A *hybrid subset space* is a triple $(X, \mathcal{O}, V)$, where $(X, \mathcal{O})$ is a subset frame and V a *hybrid valuation*, i.e., a mapping V : PROP∪NNS $\longrightarrow \mathcal{P}(X) \cup \mathcal{N}$ such that
  (a) $V(p) \subseteq \mathcal{P}(X)$ for all $p \in$ PROP,
  (b) $V(\mathfrak{n}) \in \mathcal{N}$ for all $\mathfrak{n} \in$ NNS, and
  (c) $V(\mathfrak{n}_0) = x, X$ for some $x \in X$.

$\mathcal{M} := (X, \mathcal{O}, V)$ is said to *be based on* $(X, \mathcal{O})$.

Two remarks are necessary concerning this definition. First, a hybrid valuation maps propositional variables to *sets*. As we will define the semantics of formulas with respect to *neighbourhood situations* immediately, this forces a special axiom only relevant to propositional variables. In particular, the set of validities cannot be closed under substitution therefore. Second, the special name $\mathfrak{n}_0$ really lies *between* nominals and constants: its denotation is a singleton, but only those neighbourhood situations having X as its set component are admissible.

For a given hybrid subset space we now define the satisfaction relation $\models$ between neighbourhood situations of the underlying frame and formulas in WFF. We omit the obvious clauses for the boolean connectives.

Definition 2.2 [Satisfaction and validity] Let be given a subset frame $\mathcal{S} := (X, \mathcal{O})$, a hybrid subset space $\mathcal{M} := (X, \mathcal{O}, V)$ based on it, and a neighbourhood situation x, U of $\mathcal{S}$. Then

$$
\begin{array}{lcl}
x, U \models_{\mathcal{M}} p & :\Longleftrightarrow & x \in V(p) \\
x, U \models_{\mathcal{M}} \mathfrak{n} & :\Longleftrightarrow & V(\mathfrak{n}) = x, U \\
x, U \models_{\mathcal{M}} K\alpha & :\Longleftrightarrow & y, U \models_{\mathcal{M}} \alpha \text{ for all } y \in U \\
x, U \models_{\mathcal{M}} \Box\alpha & :\Longleftrightarrow & x, U' \models_{\mathcal{M}} \alpha \text{ for all } U' \in \mathcal{O} \text{ such that } x \in U' \subseteq U \\
x, U \models_{\mathcal{M}} @_{\mathfrak{n}}\alpha & :\Longleftrightarrow & V(\mathfrak{n}) \models_{\mathcal{M}} \alpha,
\end{array}
$$

for all $p \in \mathrm{PROP}$, $\mathfrak{n} \in \mathrm{NNS}$, and $\alpha \in \mathrm{WFF}$. In case $x, U \models_{\mathcal{M}} \alpha$ is true we say that α *holds in* $\mathcal{M}$ *at* the neighbourhood situation x, U. The formula α is called *valid in* $\mathcal{M}$, iff it holds in $\mathcal{M}$ at every neighbourhood situation. Moreover, the notion of validity is extended to subset frames $\mathcal{S}$ by quantifying over all hybrid subset spaces based on $\mathcal{S}$. (Manners of writing: $\mathcal{M} \models \alpha$ and $\mathcal{S} \models \alpha$, respectively.)

As an example of hybridly valid formulas we state the correspondents of the basic properties of the set inclusion relation $\subseteq$ on $\mathcal{O}$, viz reflexivity, transitivity, and antisymmetry, making the structure $(\mathcal{O}, \subseteq)$ a partial order.

Proposition 2.3 (Properties of $\subseteq$) *Let $\mathcal{S} = (X, \mathcal{O})$ be a subset frame and $\mathfrak{n} \in \mathrm{NNS}$ a name of a neighbourhood situation. Then the following formulas are valid in $\mathcal{S}$:*

(i) $\mathfrak{n} \to \Diamond\mathfrak{n}$,

(ii) $\Diamond\Diamond\mathfrak{n} \to \Diamond\mathfrak{n}$,

(iii) $\mathfrak{n} \to \Box(\Diamond\mathfrak{n} \to \mathfrak{n})$.

3 The Logic of Hybrid Subset Spaces

In this section we provide a hybrid logical system corresponding to the class of subset spaces. To begin with, we list the bulk of the Hilbert–style axiomatization of hybrid logic given in [3], p 438 ff, which originates from [4]. Note that all axioms and rules are sound on hybrid subset spaces.

(i) $@_\mathfrak{n}(\alpha \to \beta) \to (@_\mathfrak{n}\alpha \to @_\mathfrak{n}\beta)$
(ii) $@_\mathfrak{n}\alpha \leftrightarrow \neg @_\mathfrak{n}\neg\alpha$
(iii) $\mathfrak{n} \wedge \alpha \to @_\mathfrak{n}\alpha$
(iv) $@_\mathfrak{n}\mathfrak{n}$
(v) $@_\mathfrak{n}\mathfrak{m} \wedge @_\mathfrak{m}\alpha \to @_\mathfrak{n}\alpha$
(vi) $@_\mathfrak{n}\mathfrak{m} \leftrightarrow @_\mathfrak{m}\mathfrak{n}$
(vii) $@_\mathfrak{m}@_\mathfrak{n}\alpha \leftrightarrow @_\mathfrak{n}\alpha$,

where $\mathfrak{n}, \mathfrak{m} \in$ NNS and $\alpha, \beta \in$ WFF. We argue now that the same goals can be achieved by these axioms as in usual hybrid logic. Let Γ be a maximal consistent set and $\mathfrak{n}$ a name of a neighbourhood situation. One can derive from Γ the set $\Delta_\mathfrak{n} := \{\alpha \mid @_\mathfrak{n}\alpha \in \Gamma\} \subseteq$ WFF then, which represents a natural candidate for realizing formulas of the type $@_\mathfrak{n}\alpha$ contained in Γ. The properties of $\Delta_\mathfrak{n}$ needed for this purpose can in fact be proved with the aid of the above axioms and the familiar derivation rules *modus ponens* and *necessitation;* cf [3], 7.24. In particular, $\Delta_\mathfrak{n}$ is a uniquely determined maximal consistent set containing $\mathfrak{n}$. The axiom schema called (*back*) in [3] occurs twice here because we have to take into account both modalities, K and $\Box$ (viewed as K–modalities at the moment):

(viii) $L@_\mathfrak{n}\alpha \to @_\mathfrak{n}\alpha$
(ix) $\Diamond @_\mathfrak{n}\alpha \to @_\mathfrak{n}\alpha$

The same is true for the rule (PASTE) so that we have three additional rules:

$$\text{(NAME)}\ \frac{\mathfrak{m} \to \beta}{\beta} \qquad \text{(PASTE)}_K\ \frac{@_\mathfrak{n}L\mathfrak{m} \wedge @_\mathfrak{m}\alpha \to \beta}{@_\mathfrak{n}L\alpha \to \beta}$$

$$\text{(PASTE)}_\Box\ \frac{@_\mathfrak{n}\Diamond\mathfrak{m} \wedge @_\mathfrak{m}\alpha \to \beta}{@_\mathfrak{n}\Diamond\alpha \to \beta}\,,$$

where $\alpha, \beta \in$ WFF, $\mathfrak{n}, \mathfrak{m} \in$ NNS, and $\mathfrak{m}$ is 'new' each time.

The axioms and rules stated so far enable one to perform the naming and pasting techniques of hybrid logic, respectively, in the present case as well; cf [3], p 441 ff. That is, we obtain a hybrid bimodal model $\mathcal{M}$ satisfying the following properties:

- $\mathcal{M}$ is yielded (in the sense of [3], Definition 7.26) by a maximal consistent set Γ containing the negation of a given non–derivable formula γ. In particular, every point of $\mathcal{M}$ is some $\Delta_\mathfrak{n}$, where $\mathfrak{n}$ is taken from some suitably extended set of names of neighbourhood situations. ('$\mathcal{M}$ is named'.)
- The modalities K and $\Box$ induce respective accessibility relations $\xrightarrow{L}$ and $\xrightarrow{\Diamond}$ on $\mathcal{M}$, for which the *Existence Lemma* holds; i.e., if s is a point

containing $L\alpha$, then there exists a point t such that $s \xrightarrow{L} t$ and $\alpha \in t$ (and for $\Diamond\alpha$ and $\xrightarrow{\Diamond}$ correspondingly). ('$\mathcal{M}$ is pasted'.)

- The *Truth Lemma* holds for $\mathcal{M}$; in particular, $\mathcal{M}$ falsifies γ at Γ.

We write down next the hybrid version of the usual axioms for subset spaces, cf [5] (except for the the K–axioms for K and $\Box$, which were implicitly used above already):

(x) $\mathfrak{n} \rightarrow L\mathfrak{n}$
(xi) $LL\mathfrak{n} \rightarrow L\mathfrak{n}$
(xii) $L\mathfrak{n} \rightarrow KL\mathfrak{n}$
(xiii) $(p \rightarrow \Box p) \wedge (\neg p \rightarrow \Box\neg p)$
(xiv) $\mathfrak{n} \rightarrow \Diamond\mathfrak{n}$
(xv) $\Diamond\Diamond\mathfrak{n} \rightarrow \Diamond\mathfrak{n}$
(xvi) $\Diamond L\mathfrak{n} \rightarrow L\Diamond\mathfrak{n}$,

where $p \in$ PROP and $\mathfrak{n} \in$ NNS. Note that the axiom of antisymmetry (cf Proposition 2.3 (iii)) does not occur here. (It is actually not needed later on.) Moreover, the 'different' axiom (xiii) reflects the semantics of the propositional variables, cf Definition 2.2; this axiom was already referred to following Definition 2.1. Regarding that we have axiomatized by means of *pure* formulas apart from that (and the distribution schemata), i.e., without using propositional variables, we can right state that

- the relation $\xrightarrow{L}$ is an equivalence relation,
- the relation $\xrightarrow{\Diamond}$ is reflexive and transitive, and
- the following *cross property* holds for $\xrightarrow{L}$ and $\xrightarrow{\Diamond}$: for all points s, t, u such that $s \xrightarrow{\Diamond} t \xrightarrow{L} u$, there exists a point v such that $s \xrightarrow{L} v \xrightarrow{\Diamond} u$.

We supplement the above list by an axiom having no counterpart in [5]:

(xvii) $@_{\mathfrak{n}_0} L\Diamond\mathfrak{n}$,

where $\mathfrak{n} \in$ NNS. This axiom is due to the fact that the whole space X always belongs to the opens, and is the largest one thus; cf Definition 2.1 (i). It forces the model $\mathcal{M}$ to be generated by the denotation of the special name $\mathfrak{n}_0$.

Let $[s]$ denote the $\xrightarrow{L}$–equivalence class of the point s of the model $\mathcal{M}$. We say that the class $[s]$ *precedes* the class $[t]$, and write $[s] \preccurlyeq [t]$, iff there are $s' \in [s]$ and $t' \in [t]$ such that $s' \xrightarrow{\Diamond} t'$. It is easy to see that $\preccurlyeq$ is reflexive and (due to the *cross property*) transitive. We define now five desirable properties of the relation $\preccurlyeq$. Afterwards we shall see how these properties can be established on the model $\mathcal{M}$.

Definition 3.1 [Crucial properties of $\preccurlyeq$] The relation $\preccurlyeq$ is said to be

(i) *functionally induced by* $\xrightarrow{\Diamond}$, iff, whenever $[s] \preccurlyeq [t]$, then every $s' \in [s]$ has at most one $\xrightarrow{\Diamond}$–successor contained in $[t]$,

(ii) *injectively induced by* $\stackrel{\Diamond}{\longrightarrow}$, iff, whenever $[s] \preccurlyeq [t]$, then no two distinct points $s', s'' \in [s]$ have a common $\stackrel{\Diamond}{\longrightarrow}$–successor contained in $[t]$, and

(iii) *faithfully induced by* $\stackrel{\Diamond}{\longrightarrow}$, iff, whenever $[s] \preccurlyeq [t]$, $[s] \preccurlyeq [u]$ and not $[t] \preccurlyeq [u]$, then there exists a point $s' \in [s]$ having an $\stackrel{\Diamond}{\longrightarrow}$–successor contained in $[u]$, but no $\stackrel{\Diamond}{\longrightarrow}$–successor contained in $[t]$.

(iv) Moreover, the relation $\preccurlyeq$ is said to satisfy the *factor property*, iff, whenever $[s] \preccurlyeq [t]$ and $[t] \preccurlyeq [u]$, then every $s' \in [s]$ having an $\stackrel{\Diamond}{\longrightarrow}$–successor contained in $[u]$ also has an $\stackrel{\Diamond}{\longrightarrow}$–successor contained in $[t]$.

(v) Finally, $\preccurlyeq$ is called *true*, iff, whenever $[s] \preccurlyeq [t]$ and every $s' \in [s]$ has an $\stackrel{\Diamond}{\longrightarrow}$–successor contained in $[t]$, then $[s] = [t]$.

The above properties are obviously implied by the respective first–order conditions given below. Additionally, pure correspondents are delivered by the next proposition, which are sound on hybrid subset spaces.

Proposition 3.2 (Defining these properties) *Let* $\mathcal{F} := (W, \{R, S\})$ *be a bimodal Kripke frame, where* R *corresponds to the modality* K *and* S *to* $\Box$. *Then:*

(i) $\mathcal{F} \models \Diamond\mathfrak{n} \wedge \Diamond(\mathfrak{m} \wedge L\mathfrak{n}) \rightarrow \Diamond(\mathfrak{n} \wedge \mathfrak{m})$, *iff*

$$\forall\, w, v, u \in W : (w\, S\, v \wedge w\, S\, u\, R\, v \Rightarrow v = u).$$

(ii) $\mathcal{F} \models \neg\mathfrak{n} \wedge \Diamond\mathfrak{m} \rightarrow K(\mathfrak{n} \rightarrow \neg\Diamond\mathfrak{m})$, *iff*

$$\forall\, w, v, u \in W : (w\, S\, u \wedge v\, S\, u \wedge w\, R\, v \Rightarrow w = v).$$

(iii) $\mathcal{F} \models \Diamond(\mathfrak{n} \wedge \neg L\Diamond\mathfrak{m}) \wedge L\Diamond\mathfrak{m} \rightarrow L(\Diamond L\mathfrak{m} \wedge \neg\Diamond L\mathfrak{n})$, *iff*

$$\forall\, w, v, u, t \in W : (w\, S\, v \wedge w\, R\, u\, S\, t \wedge \forall\, x : \neg(v\, R\, x\, S\, t)$$
$$\Rightarrow \exists\, s, r : (w\, R\, s\, S\, r\, R\, t \wedge \forall\, y : \neg(s\, S\, y\, R\, v))).$$

(iv) $\mathcal{F} \models \Diamond\mathfrak{m} \wedge L\Diamond(\mathfrak{n} \wedge \Diamond\mathfrak{m}) \rightarrow \Diamond(L\mathfrak{n} \wedge \Diamond\mathfrak{m})$, *iff*

$$\forall\, w, v, u, t \in W : (w\, S\, v \wedge w\, R\, u\, S\, t\, S\, v \Rightarrow \exists\, s : (w\, S\, s\, S\, v \wedge s\, R\, t))\,.$$

(v) $\mathcal{F} \models \Diamond\mathfrak{n} \wedge K\Diamond L\mathfrak{n} \rightarrow \mathfrak{n}$, *iff*

$$\forall\, w, v \in W : (w\, S\, v \wedge \forall\, u : (w\, R\, u \Rightarrow \exists\, t : u\, S\, t\, R\, v) \Rightarrow w = v)\,.$$

Proof. We exemplarily show the right–to–left direction of (ii) and the left–to–right direction of (iv). We use the terminology common in hybrid logic.

First, assume that we have $\forall\, w, v, u \in W : (w\, S\, u \wedge v\, S\, u \wedge w\, R\, v \Rightarrow w = v)$. Let $\mathcal{M}$ be an arbitrary hybrid model based on $\mathcal{F}$ and $w \in W$ a point such that $\mathcal{M}, w \models \neg\mathfrak{n} \wedge \Diamond\mathfrak{m}$. Let $u \in W$ be the denotation of $\mathfrak{m}$, i.e., $\mathcal{M}, u \models \mathfrak{m}$. Furthermore, let $v \in W$ be a point such that $w\, R\, v$ and $\mathcal{M}, v \models \mathfrak{n}$. Then we conclude $w \neq v$. It follows from the above first–oder condition that $v\, S\, u$ does not hold. Consequently, $\mathcal{M}, v \not\models \Diamond\mathfrak{m}$. This shows $\mathcal{F} \models \neg\mathfrak{n} \wedge \Diamond\mathfrak{m} \rightarrow K(\mathfrak{n} \rightarrow \neg\Diamond\mathfrak{m})$.

Now, suppose that the condition stated in (iv) is violated, i.e., $\exists\, w, v, u, t \in W : (w\, S\, v \wedge w\, R\, u\, S\, t\, S\, v \wedge \forall\, s : (w\, S\, s \Rightarrow ((s, v) \notin S \vee (s, t) \notin R)))$. Take a hybrid valuation on $\mathcal{F}$ satisfying $V(\mathfrak{n}) = \{t\}$ and $V(\mathfrak{m}) = \{v\}$. Let $\mathcal{M} := (\mathcal{F}, V)$. Then $\mathcal{M}, w \models \Diamond\mathfrak{m} \wedge L\Diamond(\mathfrak{n} \wedge \Diamond\mathfrak{m})$ and $\mathcal{M}, w \models \Box(K\neg\mathfrak{n} \vee \Box\neg\mathfrak{m})$, thus $\mathcal{F} \not\models \Diamond\mathfrak{m} \wedge L\Diamond(\mathfrak{n} \wedge \Diamond\mathfrak{m}) \rightarrow \Diamond(L\mathfrak{n} \wedge \Diamond\mathfrak{m})$. □

Thus, adding the above formulas to our system (as Axioms (xviii) – (xxii)) yields that the properties formulated in Definition 3.1 are in fact valid for $\preccurlyeq$.

Proposition 3.3 ($\preccurlyeq$ meets Definition 3.1) *The relation $\preccurlyeq$ is functionally, injectively, and faithfully induced by $\stackrel{\Diamond}{\longrightarrow}$. Moreover, $\preccurlyeq$ satisfies the factor property and is true.*

This result puts us in a position to define a hybrid subset space falsifying the given non–derivable formula γ, in the following way. Let V be the distinguished hybrid valuation of $\mathcal{M}$ and $s_0 := V(\mathfrak{n}_0)$ the denotation of the special name $\mathfrak{n}_0$ of a neighbourhood situation. Then the carrier set of the desired hybrid subset space is defined as the $\stackrel{L}{\longrightarrow}$–equivalence class of s_0 :

$$X := [s_0] .$$

Let I be the set of points t of $\mathcal{M}$ such that $[s_0]$ precedes $[t]$. Note that, given such a t, for every $s \in [s_0]$ there exists at most one $\stackrel{\Diamond}{\longrightarrow}$ –successor t_s of s contained in $[t]$, due to the functionality assertion of Proposition 3.2. Now let

$$U_t := \{s \in X \mid s \text{ has an } \stackrel{\Diamond}{\longrightarrow}\text{– successor contained in } [t]\},$$

for all $t \in I$, and $\mathcal{O} := \{U_t \mid t \in I\}$. Finally, a hybrid valuation $\widetilde{V}$ is tentatively defined on the subset frame $(X, \mathcal{O})$ by

$$\widetilde{V}(p) := X \cap V(p), \quad \text{for all } p \in \mathrm{PROP}, \text{ and}$$

$$\widetilde{V}(\mathfrak{n}) := s_{\mathfrak{n}}, U_{V(\mathfrak{n})}, \quad \text{for all } \mathfrak{n} \in \mathrm{NNS},$$

where $s_{\mathfrak{n}}$ is the $\stackrel{\Diamond}{\longrightarrow}$–predecessor of $V(\mathfrak{n})$ contained in $[s_0]$ (which is uniquely determined in case of existence, because of the injectivity assertion of Proposition 3.2).

Then we actually get that $\widetilde{\mathcal{M}} := (X, \mathcal{O}, \widetilde{V})$ bears the right structure.

Proposition 3.4 *$\widetilde{\mathcal{M}}$ is a hybrid subset space.*

Proof. We have to show that $\widetilde{V}$ is well–defined. For this purpose we apply Axiom (xvii) which says that $\mathcal{M}$ is generated by the denotation s_0 of $\mathfrak{n}_0$. (To be more precise, for all points t of $\mathcal{M}$ it holds that $(s_0, t) \in \stackrel{L}{\longrightarrow} \circ \stackrel{\Diamond}{\longrightarrow}$; note that the first–order correspondent of Axiom (xvii) is $\forall t \exists s : s_0 \, R \, s \, S \, t$.) Consequently, $[s_0]$ in fact precedes the equivalence class of the denotation of every $\mathfrak{n} \in \mathrm{NNS}$. Hence the element $s_{\mathfrak{n}}$ occurring in the definition of $\widetilde{V}$ really exists, because of the *cross property* (and is uniquely determined thus; see above). □

The decisive properties of the hybrid subset space $\widetilde{\mathcal{M}}$ are stressed by the following proposition. We use the same notations as above.

Proposition 3.5 (i) *Let $t, u \in I$. Then $[t] \preccurlyeq [u]$, iff $U_u \subseteq U_t$.*
(ii) *The relation $\preccurlyeq$ is antisymmetrical.*

Proof.

(i) First assume that $[t] \preccurlyeq [u]$. Let $s \in U_u$. Then there exists $u' \in [u]$ such that $s \xrightarrow{\diamond} u'$. According to the *factor property,* s has an $\xrightarrow{\diamond}$–prolongation to $[t]$ as well. Thus $s \in U_t$. This proves $U_u \subseteq U_t$.

Now suppose that $[t] \not\preccurlyeq [u]$. Then, because $\preccurlyeq$ is faithfully induced by $\xrightarrow{\diamond}$, there exists a point $s' \in X$ having an $\xrightarrow{\diamond}$–successor contained in $[u]$, but no $\xrightarrow{\diamond}$–successor contained in $[t]$. That is, $U_u \not\subseteq U_t$. This shows the opposite direction.

(ii) Suppose that $[t] \preccurlyeq [u]$ and $[u] \preccurlyeq [t]$. Then $\xrightarrow{\diamond}$ induces a (total) function from $[t]$ onto $[u]$, because of the *cross property* and the fact that $\preccurlyeq$ is reflexive and functionally induced by $\xrightarrow{\diamond}$. Since $\preccurlyeq$ is true, we conclude $[t] = [u]$.

$\square$

We obtain next the appropriate *Truth Lemma.*

Lemma 3.6 (Truth Lemma) *For all $\beta \in$ WFF, $s \in [s_0]$, and $t \in I$ such that t_s exists, we have*

$$s, U_t \models_{\widetilde{\mathcal{M}}} \beta \iff \beta \in t_s.$$

Proof. The proof can be done by structural induction, as usual. We only consider the case of a nominal in detail, and give some comments on the other cases afterwards.

Let $\beta = \mathfrak{n} \in$ NNS. Then,

$$\begin{aligned}
s, U_t \models_{\widetilde{\mathcal{M}}} \beta &\iff \widetilde{V}(\mathfrak{n}) = s, U_t && \text{(Def. 2.2)}\\
&\iff s = s_{\mathfrak{n}} \text{ and } U_t = U_{V(\mathfrak{n})} && \text{(Def. of } \widetilde{V})\\
&\iff s = s_{\mathfrak{n}} \text{ and } [t] = [V(\mathfrak{n})] && \text{(Prop. 3.5)}\\
&\iff s = s_{\mathfrak{n}} \text{ and } \mathfrak{n} \in V(\mathfrak{n}) = V(\mathfrak{n})_{s_{\mathfrak{n}}} = t_{s_{\mathfrak{n}}}\\
&\iff \beta = \mathfrak{n} \in t_s.
\end{aligned}$$

Note that we used once again that $\preccurlyeq$ is injectively induced by $\xrightarrow{\diamond}$, for the bottom–to–top direction of the last equivalence. This property is also applied in case $\beta = K\gamma$, where the *cross property* comes into play as well.

In case of a propositional variable Axiom (xiii) plays its part. The other boolean cases and the case of an $@_{\mathfrak{n}}$–formula, too, follow straightforwardly from the induction hypothesis. Finally, in case $\beta = \Box\gamma$ one has to take into account Proposition 3.5. $\square$

Since Γ (see above) equals t_s for some $s \in [s_0]$ and $t \in I$, Lemma 3.6 immediately gives us the following *Completeness Theorem*.

Theorem 3.7 (Completeness) *Let* $\mathbf{S}$ *be the logical system determined by the above axiom schemata and rules. Then every non–*$\mathbf{S}$*–derivable formula* γ *fails to hold at some neighbourhood situation of some hybrid subset space.*

4 Special Frame Classes

It is not hard to get now hybrid completeness results for several interesting classes of subset frames, since the additional properties defining these classes can actually be expressed by pure formulas.

Let us start with *linear* frames, where $\mathcal{O}$ forms a descending chain with respect to $\subseteq$ by definition. The importance of this class of structures results from the fact that they support modelling of temporally evolving sets. Examples of such sets we have in mind are the actual knowledge state of an agent, and point sets representing geometric objects which change in the course of time.[2] The problem of axiomatizing linear frames was our original motivation to treat subset spaces hybridly. For, no corresponding axiomatization is known with regard to the originally modal approach to subset spaces in [5]. Hybrid logic, however, is a great help in that. In fact, it turns out that linearity corresponds to *dichotomy* of the composite relation $\xrightarrow{L} \circ \xrightarrow{\Diamond}$. This property can be defined by the schema

(xxiii) $@_{\mathfrak{m}} L \Diamond \mathfrak{n} \vee @_{\mathfrak{n}} L \Diamond \mathfrak{m}$.

Thus we need only add this axiom to the list of the previous section in order to obtain completeness. But we can even cross out Axiom (xvii) then, since Axiom (xxiii) implies a chain structure of the $\xrightarrow{L}$–equivalence classes on the canonical model (i.e., the above model $\mathcal{M}$) with respect to the relation $\preccurlyeq$. Therefore, we can take advantage of a suitably generated submodel of $\mathcal{M}$ then.[3]

We will refer to linear subset spaces once again at the end of this section, in connection with effectivity.

Axiom (xvii) can be omitted also in case of *treelike* spaces. A subset frame $(X, \mathcal{O})$ satisfies this property by definition, iff

$$\forall U, V \in \mathcal{O} : (U \cap V \neq \emptyset \Rightarrow (U \subseteq V \vee V \subseteq U)) .$$

Such tree structures play an important part in modelling the development of knowledge in case of branching time semantics; see [7], where treelike spaces were studied 'modally'. Presently, only the hybrid correspondent to *weak*

[2] Thus a connection opens up here between topological and (spatio–)temporal reasoning.
[3] It should be remarked that a different axiomatization of linear subset spaces was given in [10], where the subsequent Axiom (xxiv) was involved in (among others); see also [11].

connectedness has to be added to the axiom list from Section 3:

(xxiv) $\Diamond\mathfrak{n} \wedge \Diamond\mathfrak{m} \rightarrow \Diamond\left((\mathfrak{n} \wedge \Diamond\mathfrak{m}) \vee (\mathfrak{m} \wedge \Diamond\mathfrak{n})\right)$

The completeness results just indicated, and a couple of others, are contained in summary in the following theorem.

Theorem 4.1 (Some axiomatizable classes of subset spaces) *A hybrid logic is yielded which is complete for the respective class of frames, by adding the following axioms to the list from Section 3:*

(i) $@_{\mathfrak{m}} L\Diamond\mathfrak{n} \vee @_{\mathfrak{n}} L\Diamond\mathfrak{m}$ *in case of linear subset spaces;*

(ii) $\Diamond\mathfrak{n} \wedge \Diamond\mathfrak{m} \rightarrow \Diamond\left((\mathfrak{n} \wedge \Diamond\mathfrak{m}) \vee (\mathfrak{m} \wedge \Diamond\mathfrak{n})\right)$ *in case of treelike spaces;*

(iii) $\Diamond\mathfrak{n} \wedge L\Diamond\mathfrak{m} \rightarrow \Diamond\left(\Diamond\mathfrak{n} \wedge L\Diamond\mathfrak{m} \wedge K\Diamond L(\mathfrak{n} \vee \mathfrak{m})\right)$ *in case of* union–closed *subset spaces (i.e., spaces satisfying* $U, V \in \mathcal{O} \Rightarrow U \cup V \in \mathcal{O}$*).*

(iv) $\Diamond\mathfrak{n} \rightarrow \mathfrak{n}$ *in case of* trivial *spaces (where* $\mathcal{O} = \{X\}$*).*

Note that item (iv) of this theorem can easily be generalized to spaces where chains of subsets have a fixed finite length, in particular, *flat cpo*s. The hybrid correspondent in the latter case is $\Diamond(\mathfrak{n} \wedge \Diamond\mathfrak{m}) \rightarrow \Diamond(\mathfrak{n} \wedge \mathfrak{m}) \vee \mathfrak{n}$.

What is more, the hybrid method also works for subset spaces with respect to the *strict* inclusion relation $\subset$. In fact, strictness corresponds to *irreflexivity* of $\stackrel{\Diamond}{\longrightarrow}$, which can be defined by a pure formula; cf [3]. Then, not only linearity, but also ramification of prescribed degree of the relation $\subset$ can be expressed. (By the way, linearity corresponds to *trichotomy* of $\stackrel{L}{\longrightarrow} \circ \stackrel{\Diamond}{\longrightarrow}$ now.) Moreover, we can also handle *density* of the set inclusion relation in the strict case, which corresponds to $\Diamond\mathfrak{n} \rightarrow \Diamond\Diamond\mathfrak{n}$. This goes beyond the purely modal possibilities up to now, too. (Details concerning dense subset spaces will appear elsewhere.)

We stop discussing completeness here. Concluding this section, we touch on effectivity properties of the hybrid systems considered in this paper. We confine ourselves to the logic **L** of linear frames though, for which we obtain the smallest possible complexity bound. This may be interpreted as an indication of reasonable preconditions for practical applications the system offers.

Theorem 4.2 (Complexity) *The set of* **L***–satisfiable formulas is complete in* NP*.*

For a proof of the theorem certain model decomposition techniques can be modified appropriately, which were developed for treelike spaces in [7]; see also [12], where this method was applied to the case of sets *increasing* in the course of time. We focus on the main steps of the proof of Theorem 4.2 in the following, introduce the relevant notions, and give reasons for the the validity of the crucial intermediary assertions.

Definition 4.3 [Segmentation of a linear order] Let $\mathcal{T} := (T, \leq)$ be a (reflexive) linear order.

(i) A subset $\emptyset \neq T' \subseteq T$ is called a *segment* of $\mathcal{T}$, iff there is no $t \in T \setminus T'$ strictly between any two elements of T'.

(ii) A partition of T into segments is called a *segmentation* of $\mathcal{T}$.

Given a linear hybrid subset space $\mathcal{M} = (X, \mathcal{O}, V)$ and a formula α, we will have to consider segmentations of the linear order $(\mathcal{O}, \subseteq)$ [4] such that the truth value of α remains unaltered on every segment.

Definition 4.4 [Stability] Let $\alpha \in \mathrm{WFF}$ be a formula, $\mathcal{M} = (X, \mathcal{O}, V)$ a linear hybrid subset space, I an indexing set and $\mathcal{P} := \{O_\iota \mid \iota \in I\}$ a segmentation of $\mathcal{O}$. Then α is called *stable on* $\mathcal{P}$, iff for all $\iota \in I$ and $x \in X$ we have either

$$x, U \models_{\mathcal{M}} \alpha \quad \text{for all } U \in O_\iota \text{ such that } x \in U, \text{ or}$$

$$x, U \models_{\mathcal{M}} \neg\alpha \text{ for all } U \in O_\iota \text{ such that } x \in U.$$

It turns out that we can always achieve a *finite* segmentation of $\mathcal{O}$ on which a given formula is stable.

Proposition 4.5 *Let $\alpha \in \mathrm{WFF}$ be a formula and $\mathcal{M} = (X, \mathcal{O}, V)$ a linear hybrid subset space. Then there exists a finite segmentation $\mathcal{P}_\alpha = \{O_1, \ldots, O_n\}$ of $\mathcal{O}$ such that α is stable on $\mathcal{P}_\alpha$. Moreover, $\mathcal{P}_\alpha$ can be chosen in such a way that it refines $\mathcal{P}_\beta$ for every subformula β of α.*

Proof. The partition $\mathcal{P}_\alpha$ is constructed by induction on the structure of α, starting with the trivial segmentation $\{\mathcal{O}\}$ in case of a propositional variable p. Note that Axiom (xiii) guarantees stability of p on $\{\mathcal{O}\}$.

In case $\alpha = \mathfrak{n} \in \mathrm{NNS}$ we define $\mathcal{P}_\alpha := \{\{U' \in \mathcal{O} \mid U' \supset U\}, \{U\}, \{U' \in \mathcal{O} \mid U' \subset U\}\}$, where U is determined by $V(\mathfrak{n}) = x, U$ (for some $x \in X$). Clearly, $\mathfrak{n}$ is stable on $\mathcal{P}_\alpha$.

As to the other boolean cases and the two modal ones, respectively, see [12], proof of Proposition 13.

Finally, in case $\alpha = @_{\mathfrak{n}}\beta$ we can evidently choose $\mathcal{P}_\alpha := \mathcal{P}_\beta$. □

Let be given a formula $\alpha \in \mathrm{WFF}$ and a linear hybrid subset space $\mathcal{M} = (X, \mathcal{O}, V)$. We consider a finite segmentation $\mathcal{P}_\alpha = \{O_1, \ldots, O_n\}$ of $\mathcal{O}$ according to Proposition 4.5; note for later purposes that n depends on α. We define

$$U_i := \bigcup_{U \in O_i} U,$$

for all $i \in \{1, \ldots, n\}$, and $\mathcal{M}' := (X, \{U_1, \ldots, U_n\}, V)$. Then $\mathcal{M}'$ is obviously a linear hybrid subset space. Moreover, $\mathcal{M}'$ is semantically equivalent to $\mathcal{M}$ with regard to α. This is the content of the next proposition.

Proposition 4.6 *For all subformulas β of α, $x \in X$, $i \in \{1, \ldots, n\}$ and $U \in O_i$ such that $x \in U$, we have*

$$x, U \models_{\mathcal{M}} \beta \iff x, U_i \models_{\mathcal{M}'} \beta.$$

[4] For this special linear order, we do not write down '$\subseteq$' any longer.

Proof. By induction on α; see [12], Proposition 14. (The hybrid cases are trivial.) □

In particular, passing from $\mathcal{M}$ to $\mathcal{M}'$ preserves satisfiability of α. We have, therefore, restricted the problem of deciding whether a given formula is satisfiable on a linear hybrid subset space, to the class of such models having 'finite depth of the set of opens' additionally. This is the key step towards decidability of **L**. But we wanted even more, viz NP–completeness. To this end we have to ensure that the number n of segments depending on α (see Proposition 4.5 and the remark after it) does not increase too fast. And this can actually be achieved.

Lemma 4.7 *In Proposition 4.5,* $(\text{length}(\alpha))^2$ *can be chosen as an upper bound for* n*.*

Proof. Consider the parse tree of the formula α. The construction of the segmentation $\mathcal{P}_\alpha$ starts at the leaves of this tree. What can happen on the way to the root? Well, only in case '$\wedge$' or 'K' is attributed to an inner node the partition obtained so far has to be refined. In case of a conjunction the resulting number of segments is essentially the sum of the number of segments associated to each of the conjuncts, and this does not cause any problems. However, in case of an application of the knowledge operator every segment could be cut in half. So, if the K–rank of α is too big we cannot even get a polynomial bound for n at first glance.

But the S5–properties of K prove to be helpful now. Accordingly, a whole chain of unary operators (containing at least one 'K') has the same effect on a partition into segments as a single K–operator. Thus, it suffices to consider the *reduced parse tree* of α, where every unary ramified node is the daughter of a binary ramified one.

It is then immediately clear that we would get the desired bound if the reduced parse tree $\mathfrak{T}$ of α would be 'full', because a $\mathfrak{T}$ like that satisfies $\text{depth}(\mathfrak{T}) \leq 2 \cdot \log \text{length}(\alpha)$, which implies $n \leq 2^{2 \cdot \log \text{length}(\alpha)} = (\text{length}(\alpha))^2$ (up to a constant factor).

The general case requires a more careful estimate. Let l be the number of segments occurring 'on the left and below' K, and r the corresponding number of segments on the right–hand side. Then, the resulting number of new segments is really less than or equal to $(l + r - 1) + \min\{l, r\}$. This yields by induction $n \leq m^2$, where m is the number of nodes of $\mathfrak{T}$. Because of that the lemma is proved in this case as well. □

With the aid of Lemma 4.7 we obtain that the complexity of **L** is no worse than that of its sublogic S4.3; cf [3], Theorem 6.41. Note that adding nominals and corresponding satisfaction operators does not alter the complexity in case of the basic modal logic either; cf [1].

5 Concluding Remarks

We have studied the basic hybrid theory of subset spaces in this paper. We obtained 'canonical' completeness for the most general system, i.e., the logic of arbitrary subset spaces. Further completeness results for more special frame classes turned out to be rather simple extensions of this particular one. However, the above collection is surely not exhaustive. What about a hybrid treatment of *topological* and *intersection–closed* spaces, respectively? This would be especially interesting, in comparison with [5] and [14].

The investigation of extensions and variations of the underlying hybrid language is a demanding task, too. For instance, what about the integration of state variables and the $\downarrow$–binder? Or, how can the different approach to hybridizing logics of subset spaces be carried out satisfactorily, which was mentioned in the introduction (that one including both 'real' nominals and names of sets)?

These questions should be answered by future research. Furthermore, we touched on the question of practical applicability of our logic in the introduction. The development of implementation–oriented calculi like tableaux is a necessary precondition for this purpose, which in fact seems to be promising; cf [2] for a closer justification.

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