On Hybrid Arrow Logic

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Abstract

We investigate hybridization of Arrow Logic. Hybridization is a general technique for increasing the power of modal logics. A hybrid extension of a given modal logic can be obtained in two stages: first adding a new sort of variables considered as formulas whose intended meaning is to denote points in the domains of the frames then introducing mechanisms for handling the now available information (e.g. the satisfiability operator @ and the down-arrow binder $\downarrow$). The hybrid language $\mathcal{H}(\downarrow, @)$, obtained from the basic modal language following this construction, may be regarded as the "standard hybrid formalism". In this paper, we study a system obtained from a hybridization of Arrow Logic much as $\mathcal{H}(\downarrow, @)$ was obtained from the basic modal language.

1 Introduction

We investigate two approaches to hybridizing Arrow Logic (AL). Hybridization is a general technique for increasing the power of modal logics [1,4]: any modal logic may be 'hybridized' [7]. A hybrid extension of a given modal logic can be built in two stages. The first adds to the language a new sort of variables that will be *formulas* used to *denote* points in the domains of the frames. The second introduces mechanisms for handling additional information now available. There is a wide range of possibilities for the second step [6,7], the *satisfiability operator* @ [3] and the *down-arrow binder* $\downarrow$ [6,7] being the more intensively studied ones. The hybrid language $\mathcal{H}(\downarrow, @)$, obtained from the basic modal language by introducing these two mechanisms, may be regarded as a kind of *standard hybrid formalism* [2]. In this paper, we study a system obtained hybridizing AL much as $\mathcal{H}(\downarrow, @)$ was obtained from the basic modal language.

AL is a system for formal reasoning about objects that may be viewed as arrows [19]. There are different semantics for AL, depending on the ontology chosen for arrows. In the *abstract semantics*, arrows are not assumed to have any explicit internal structure. In this case, a frame consists of a set A of objects, called *arrows*, together with three relations C, R, I, called, *composition*, *reversion*, and *identity*, respectively, following the natural intuition (for instance, Rxy means that y is the *reverse* of x). AL under the abstract semantics is a generalization of the basic modal language when some more evolved modal operators are considered [5].

In a more concrete semantics, which we call *square*, arrows, instead of being primitive entities, are ordered pairs of more fundamental objects. In this case, a frame consists of the set $\mathcal{U} \times \mathcal{U}$ of the ordered pairs of *points* in $\mathcal{U}$ and the three relations C, R, I are defined according to the internal structure of arrows (for instance, $R(a, b)(c, d)$ iff $a = d$ and $b = c$). The class of square frames is denoted by SQ. Observe that in this setting propositional variables denote sets of ordered pairs; so, AL under Square Semantics (AL2) is a kind of modal relational formalism. In [14,16,17,20] the close relationship between AL2 and the relational formalism of Relation Algebras (RA) [13,18] was investigated. AL2 is equipollent in means of expression and proof to "the three variable fragment of first-order logic". This shows that AL2 is an interesting modal alternative to RA. Y. Venema developed an extension of AL2 by a non-orthodox rule (a version of the irreflexivity rule [12] for the difference operator [11]) that is strongly sound and complete with respect to SQ [20].

Since AL2 has limited expressive power and the above extension seems to be not very adequate for proving theorems, two problems related to AL2 arise: extend AL2 towards achieving equipollence with the First-Order Logic (FOL) of binary relations in means of expression (through new operators) and of proof (through a set of adequate rules).

A relation algebraic formalism, Relation Algebra with Binders (RAB), was introduced [15] to tackle these two problems. RAB is a hybrid-like extension of RA obtained by introducing the variables $x, y, z \dots$ and the $\downarrow$ to bind them. RAB was proved to be equipollent with the FOL of binary relations in means of expression and semantic consequence [15]. Axiomatization and adequacy of RAB is investigated elsewhere [10].

Here we obtain another hybridization of AL2, called Two-Dimensional Hybrid Arrow Logic (HAL2), following a very similar but different route.

A motivation for the hybridization of RA comes from the symbolization of statements about binary relations. For this purpose, a more adequate alternative than FOL is the language obtained from the relational algebraic one by introducing some hybrid machinery, as argued in [15].

RAB is defined by extending the relation algebraic language with variables $x, y, z, \dots$ to denote the relations $\{(a, a)\}$, together with operators $\downarrow_x$, to form the term $\downarrow_x R$ from the relational term R. The evaluation of $\downarrow_x R$ in a pair (a, b) is: *denote the* $\{(a, a)\}$ *relation by* x *and evaluate the term* R *in* (a, b).

In this system we can symbolize any elementary statement about binary relations. As an example, consider the 'sibling' relation: *siblings are two persons* **that** *share both parents*. Let the relations 'is a sibling of' and 'is a parent of' be symbolized by S and P, respectively. In this language we have the following symbolization for the sibling relation:

$$S := \neg\iota\delta\wedge \downarrow_x (\otimes(P \wedge ((\neg\iota\delta \wedge (P \circ x \circ \otimes P)) \circ P)) \circ \top)$$

An alternative symbolization using pair-variables emerges when we realize that in the definition of the sibling relation the relative pronoun **that** refers to a pair of individuals:

$$S := \neg\iota\delta\wedge \downarrow_{(x,y)} (\otimes P \circ (\neg\iota\delta \wedge (P \circ (y, x) \circ \otimes P)) \circ P)$$

In the first symbolization, x denotes the relation $\{(a, a)\}$, referring to the point a indirectly, whereas in the second, the pair (x, y) denotes directly the arrow (a, b), rather than its coordinates. Also, in the first symbolization, $\downarrow_x$ links x to a, while, in the second, $\downarrow_{(x,y)}$ links (x, y) to (a, b).

In HAL2 we introduce variables that denote directly the pairs rather than their coordinates. In this manner, we are following closely the hybridization paradigm, introducing variables to denote the 'first-class citizens' of the frames. Second, in RAB the binder apparatus handles the available new information by linking variables directly to points, whereas in HAL2 the binder links pair-variables directly to pairs. So, the new information is manipulated directly by binding pairs.

To make the hybridization of AL2 smoother, we proceed in two steps. First, we define an intermediate hybrid extension of AL, called Hybrid Arrow Logic (HAL). HAL is obtained from AL under abstract semantics, following directly the standards of hybridization. In particular, we present an axiomatization of this system by extending the results in [8]. Second, we define HAL2 by making two essential modifications in HAL: introducing *pair-variables* and restricting the semantics of HAL to squares under a subclass of assignments. Pair-variables are of the form (x, y), where x and y are symbols from a basic set. Since these new variables are viewed as ordered pairs, the set of pair-variables can be viewed as a square in a natural way. When restricting the semantics of HAL to squares we must also restrict satisfaction to a subclass of assignments that respect the two-dimensional structure of these variables. This framework provides a formalism appropriate for talking about binary relations: we can express all the elementary statements about them and obtain modularly an adequate axiom system for proving theorems.

This paper is organized as follows. In Section 2, we hybridize AL under abstract semantics, following the standard hybridization paradigm [2]: we introduce variables to denote arrows as well as the satisfaction operator and the down-arrow binder to manipulate the new information obtained. This section includes a strongly sound and complete axiomatization for the system here introduced, obtained by extending the techniques presented in [8]. In Section 3, our approach to hybridizing AL2 is presented: HAL2. The main features of our approach are the use of pair-variables for referring to arrows directly and the use of the structure underling these variables to obtain an axiomatization of this new system. In Section 4, we present a summary of the system in [15] (RAB) in a modal setting and compare it with the approach presented in Section 3 (HAL2). Given the close connection between RA and AL, it is natural to study the relationship between RAB and HAL2. We prove that these two systems are equivalent, thereby obtaining an axiomatization of RAB (and solving in a modular way a problem proposed in [15]). Finally, Section 5 concludes with some comments and proposals for future work.

2 Hybrid Arrow Logic

Hybrid Arrow Logic (HAL) is an extension of AL through the standard process of hybridization [1,4,6,7].

The *alphabet* of HAL is that of AL plus the *variables* $X_i : i \in \mathbb{N}$, the *satisfaction operator* @ and the *down-arrow operator* $\downarrow$. The set of variables is denoted by VAR and X, Y, Z denote arbitrary variables. The *hybrid arrow formulas* are obtained by closing the set of formulas of AL by the rules: $\alpha := X \mid @_X\alpha \mid\ \downarrow_X \alpha$. Frames, models and rooted models of HAL are as in AL.

Semantics of the hybrid apparatus is given by *assignments* $G : \mathsf{VAR} \to S$ mapping variables to arrows in an arrow frame $\mathcal{F} = \langle S, C, R, I\rangle$. As in FOL, we define the update $(A/X)G$ of an assignment G to be the assignment that agree with G in all variables but X, which is mapped to A. The *satisfaction* of a formula α in a rooted arrow model $\langle \mathcal{M}, A\rangle$ (with $\mathcal{M} = \langle S, C, R, I, V\rangle$) under assingment G (denoted by $\mathcal{M}, A, G \models \alpha$) is defined as in AL, with the following extra clauses:

1. $\mathcal{M}, A, G \models X$ iff $G(X) = A$.
2. $\mathcal{M}, A, G \models @_X\alpha$ iff $\mathcal{M}, G(X), G \models \alpha$.
3. $\mathcal{M}, A, G \models \downarrow_X \alpha$ iff $\mathcal{M}, A, (A/X)G \models \alpha$.

A formula α is a *consequence* of a set of formulas Γ in HAL, denoted by $\Gamma \models_{\mathrm{HAL}} \alpha$, when $\mathcal{M}, A, G \models \Gamma$ implies $\mathcal{M}, A, G \models \alpha$, for every rooted arrow model $\langle \mathcal{M}, A\rangle$ and every assignment G.

Substitution of variables for variables in formulas is as usual, guarding against accidental binding. The set of free variables of a formula α, denoted by $\mathsf{free}(\alpha)$, and the notion of a variable being *substitutable* for another in a given formula are as in FOL, considering that the down-arrow operator $\downarrow$ binds variables and the satisfaction operator does not. Thus the following result holds:

Lemma 2.1 (Substitution) *If Y is substitutable for X in α, then we have* *that* $\mathcal{M}, A, (G(Y)/X)G \models \alpha$ *iff* $\mathcal{M}, A, G \models (Y/X)\alpha$.

An axiom system for HAL is obtained by joining the axiom system for AL [19] with the axiom system for hybrid logic [8].

Normal) *Axioms and rules for a normal extended modal logic.*
AL) *Axioms for arrow logic* (as in [19] [1]).
Name) $@_X X$.
Nom) $@_X Y \to (@_X\alpha \to @_Y\alpha)$.
Elimination) $X \wedge @_X\alpha \to \alpha$.
Scope) $@_X @_Y\alpha \leftrightarrow @_Y\alpha$.
Self-Dual @) $@_X\alpha \leftrightarrow \neg @_X \neg\alpha$.
Self-Dual ↓) $\downarrow_X \alpha \leftrightarrow \neg \downarrow_X \neg\alpha$.
Bridge ∘) $@_X(Y \circ Z) \wedge @_Y\alpha \wedge @_Z\beta \to @_X(\alpha \circ \beta)$.
Bridge ⊗) $@_X \otimes Y \wedge @_Y\alpha \to @_X \otimes \alpha$.
Q1) $\downarrow_X \alpha \leftrightarrow \alpha$, if $X \notin \mathsf{free}(\alpha)$.
Q2) $\downarrow_X \alpha \to (Y \to (Y/X)\alpha)$, if Y is substitutable for X in α.
Q3) $\downarrow_X (X \to \alpha) \to \downarrow_X \alpha$.

Paste ⊗) $\dfrac{@_X \otimes Y \wedge @_Y\alpha \to \beta}{@_X \otimes \alpha \to \beta}$, if Y is new.

Paste ∘) $\dfrac{@_X(Y \circ Z) \wedge @_Y\alpha \wedge @_Z\beta \to \gamma}{@_X(\alpha \circ \beta) \to \gamma}$, if Y and Z are new.

Normal (that includes K axioms and necessitation rules for the duals of hybrid operators) states that HAL is a normal modal logic (which assures, for instance, Substitution of Equivalents [5]). Name, Nom, Elimination and Scope explain that variables denote arrows. Self-Dual express that the hybrid operators are self-duals. Axioms Bridge and rules Paste are particularizations, for the multi-modal case, of axioms and rules of hybrid logic [8,9]. Rules Paste act as the reverse of the axioms Bridge; together, they internalize the satisfaction conditions of the Peircean operators. Q1, Q2 and Q3 ensure that the down-arrow operator behaves as a quantifier.

We write $\Gamma \vdash_{\mathrm{HAL}} \alpha$ to express that α is derivable from Γ, in the HAL system. The notion of derivability is defined as usual, with the rules of necessitation and Paste restricted to *theorems* (i.e., formulas derived from the empty set).

Let $[\gamma/\beta]\alpha$ denote the result of replacing zero, one or more occurrences of the formula γ by the formula β in the formula α.

Lemma 2.2 (Substitution of Equivalents – SEq) *If* $\vdash_{\mathrm{HAL}} \gamma \leftrightarrow \beta$, *then* $\vdash_{\mathrm{HAL}} [\gamma/\beta]\alpha \leftrightarrow \alpha$.

Property **SEq** is a useful tool for proving theorems within the system.

[1] For instance, the axioms $\alpha \to \otimes \otimes \alpha$, $\alpha \to \iota\delta \circ \alpha$, and $\alpha \circ \neg(\otimes\alpha \circ \beta) \to \neg\beta$.

Proposition 2.3 *The following are theorems of* HAL*:*
Negation) $@_X \neg\alpha \leftrightarrow \neg @_X \alpha$.
Conjunction) $@_X(\alpha \wedge \beta) \leftrightarrow @_X\alpha \wedge @_X\beta$.
Introduction) $X \wedge \alpha \rightarrow @_X\alpha$.
Swap) $@_X Y \leftrightarrow @_Y X$.
Bridge $\downarrow$*)* $@_X(X/Y)\alpha \leftrightarrow @_X \downarrow_Y \alpha$*, if* X *is substitutable for* Y *in* α.

Proof. By Normal, Name, Nom, Elimination, Self-Dual @, Q2, and **SEq**. □

It follows from Q1, *Bridge* $\downarrow$, and **SEq** that in HAL we also have a notion of alphabetic variant, as in FOL.

We shall now prove the Completeness Theorem for HAL, i.e., that $\models_{\text{HAL}} \subseteq \vdash_{\text{HAL}}$. We use the canonical model construction, together with the technique of extending consistent sets to consistent sets with witnesses. This mixture is usual in hybrid logic [8,9]. Our proof uses an extension of the techniques presented in [8].

We start with the canonical model $\mathcal{M}^C = \langle S^C, C^C, R^C, I^C, V^C\rangle$ of HAL [5]. Since we have axiom AL, model $\mathcal{M}^C$ is an arrow model. But we do not have a natural way to define an assignment to variables in $\mathcal{M}^C$. So we work with a substructure $\mathcal{M}^\Gamma$ of $\mathcal{M}^C$, defined by a consistent set Γ, for which we can define a natural assignment G^Γ to the variables. When Γ has certain characteristics, we can use $\mathcal{M}^\Gamma$ to prove the Model Existence Theorem. Thus we prove an extended Lindenbaum lemma, showing that every consistent set can be extended to one with these characteristics. This will finish our completeness proof.

A set of formulas Γ is *named* if $X \in \Gamma$, for some $X \in \mathsf{VAR}$. Γ has *witnesses* if for each $X \in \mathsf{VAR}$ and formula α, $@_X \otimes \alpha \in \Gamma$ implies $@_X \otimes Y \wedge @_Y\alpha \in \Gamma$, for $Y \in \mathsf{VAR}$, and $@_X(\alpha \circ \beta) \in \Gamma$ implies $@_X(Y \circ Z) \wedge @_Y\alpha \wedge @_Z\beta \in \Gamma$, for $Y, Z \in \mathsf{VAR}$.

Given a maximal consistent set (MCS) Γ, define $\Delta_X^\Gamma = \{\alpha : @_X\alpha \in \Gamma\}$, for each $X \in \mathsf{VAR}$.

Lemma 2.4 *If* Γ *is an MCS, then* Δ_X^Γ *is an MCS, for all* $X \in \mathsf{VAR}$.

Proof. By *Negation* and *Conjunction.* □

Lemma 2.5 *If* Γ *is an MCS and* $X, Y \in \mathsf{VAR}$*, then* $X \in \Delta_Y^\Gamma$ *iff* $\Delta_X^\Gamma = \Delta_Y^\Gamma$.

Proof. By Name, Nom, and *Swap.* □

The *model yielded by* an MCS Γ is $\mathcal{M}^\Gamma = \langle S^\Gamma, C^\Gamma, R^\Gamma, I^\Gamma, V^\Gamma\rangle$, the substructure of the canonical model where $S^\Gamma = \{\Delta_X^\Gamma : X \in \mathsf{VAR}\}$. The *natural assignment* in $\mathcal{M}^\Gamma$ is $G^\Gamma : \mathsf{VAR} \to S^\Gamma$ such that $G^\Gamma(X) = \Delta_X^\Gamma$.

Lemma 2.6 *If* Γ *is an MCS, then* $\otimes Y \in \Delta_X^\Gamma$ *iff* $R^\Gamma(\Delta_X^\Gamma, \Delta_Y^\Gamma)$*, and* $Y \circ Z \in \Delta_X^\Gamma$ *iff* $C^\Gamma(\Delta_X^\Gamma, \Delta_Y^\Gamma, \Delta_Z^\Gamma)$.

Proof. By Name, Bridge $\otimes$, and Bridge $\circ$. □

The model $\mathcal{M}^\Gamma$ yielded by MCS Γ with witnesses is an arrow frame. To verify this fact, since $\mathcal{M}^\Gamma$ is a substructure of an arrow model, it suffices to prove the following.

Lemma 2.7 *If Γ is an MCS with witnesses, then for each $X \in \mathsf{VAR}$ we have $R^\Gamma(\Delta^\Gamma_X, \Delta^\Gamma_Y)$, $C^\Gamma(\Delta^\Gamma_X, \Delta^\Gamma_Z, \Delta^\Gamma_X)$, and $I^\Gamma(\Delta^\Gamma_Z)$, for some $Y, Z \in \mathsf{VAR}$.*

Proof. By Name, Lemma 2.6, and the following instances of AL: $X \leftrightarrow \otimes\otimes X$ and $X \to \iota\delta \circ X$. □

We use the model yield by Γ to prove the Model Existence Theorem: if Γ is a named MCS with witnesses, then $\mathcal{M}^\Gamma, A, G^\Gamma \models \Gamma$, for some $A \in S^\Gamma$. The following two lemmas will determine this $A \in S^\Gamma$.

Lemma 2.8 (Satisfiability Lemma) *If Γ has witnesses, then $\alpha \in \Delta^\Gamma_X$ iff $\mathcal{M}^\Gamma, \Delta^\Gamma_X, G^\Gamma \models \alpha$.*

Proof. By induction on the structure of α. The Boolean cases are straightforward, the Peircean cases follow from the assumption that Γ has witnesses, the variable case follows from Lemma 2.5, the satisfaction operator case follows from Scope, the down-arrow operator case follows from *Bridge* $\downarrow$ and the Substitution Lemma 2.1, using alphabetic variants. □

Lemma 2.9 *If Γ is named, then $\Gamma = \Delta^\Gamma_X$, for some $X \in \mathsf{VAR}$.*

Proof. By $\mathsf{Elimination}$ and Lemma 2.4. □

Now we will show that every consistent set of formulas can be extended to an MCS named and with witnesses, as required by Lemmas 2.8 and 2.9. Let HAL^N be the system HAL with the language enriched with a countable set of new variables NVAR.

Lemma 2.10 (Extended Lindenbaum Lemma) *Every consistent set in* HAL *can be extended to a named MCS with witnesses in* HAL^N*.*

Proof. Let $\Gamma_0 = \Gamma \cup \{X_0\}$ with X_0 being the first new variable in an enumeration of NVAR. Γ_0 is consistent, from axioms $\mathsf{Q1}$ and $\mathsf{Q3}$. Let $\alpha_0, \alpha_1, \alpha_2, \ldots$ be an enumeration of the formulas of HAL^N. Let Θ_0 be Γ_0. For $m \geq 1$, let Θ_{m+1} be Θ_m, when $\Theta_m \cup \{\alpha_m\}$ is inconsistent, and $\Theta_m \cup \{\alpha_m\} \cup \Sigma$ otherwise ($\Sigma = \emptyset$ if α_m is not of the form $@_X \otimes \alpha$ or $@_X(\alpha \circ \beta)$; $\Sigma = \{@_X \otimes \alpha, @_X \otimes Y, @_Y \alpha\}$ if α_m is $@_X \otimes \alpha$, Y being the first variable in NVAR not occurring in Θ_m neither in $@_X \otimes \alpha$; $\Sigma = \{@_X(\alpha \circ \beta), @_X(Y \circ Z), @_Y \alpha, @_Z \beta\}$ if α_m is $@_X(\alpha \circ \beta)$, Y and Z being the first variables in NVAR not occurring in Θ_m neither in $@_X(\alpha \circ \beta)$). Let $\Theta = \bigcup_{n \geq 0} \Theta_n$. By rules $\mathsf{Paste}\ \otimes$ and $\mathsf{Paste}\ \circ$, Θ is consistent. □

Lemmas 2.8, 2.10, and 2.9 yield Model Existence Theorem and, as a corollary, HAL completeness.

Theorem 2.11 (Model Existence) *Every consistent set of* HAL *formulas is satisfied in a rooted arrow model with some assignment.*

Theorem 2.12 (Completeness) *If* $\Gamma \models_{\text{HAL}} \alpha$*, then* $\Gamma \vdash_{\text{HAL}} \alpha$.

3 Two-dimensional Hybrid Arrow Logic

In this section we present *Two-dimensional Hybrid Arrow Logic* (HAL2) as an hybridization of AL2. This hybridization considers the two-dimensional aspect of the square semantics. In the two-dimensional semantics of AL, arrows have first and second coordinates. This feature is captured, in the syntax of HAL2, by the introduction of pair-variables and, in its semantics, by the restriction to assignments that respect their two-dimensionality.

HAL2 is a version of HAL in which the set of variables VAR is the Cartesian product $\mathsf{Var}\times\mathsf{Var}$, restricted to the square semantics and to a special class of assignments. Arbitrary elements of Var are denoted by x, y, z. The *two-dimensional arrow formulas* are obtained by closing the set of formulas of AL by the rules: $\alpha := (x, y) \mid @_{(x,y)}\alpha \mid \downarrow_{(x,y)} \alpha$.

To interpret the pair-variables, the satisfaction operator and the binder as in hybrid logic, we need assignments of arrows to the pair-variables. An intuitive way to do this is to define assignments as functions mapping pair-variables to arrows. To obtain an expressive hybrid square arrow logic, however, we need a more refined semantics.

Let $\mathcal{F} = \langle \mathcal{U} \times \mathcal{U}, C, R, I\rangle$ be a square frame. Given an assignment $G : \mathsf{Var}\times\mathsf{Var} \to \mathcal{U}\times\mathcal{U}$, the function $g_G : \mathsf{Var} \to \mathcal{U}$ is defined by $g_G(x) = \Pi_1(G(x, x))$, where Π_1 gives the first coordinate of the pair $G(x, x)$, for each $x \in \mathsf{Var}$. Given a function $g : \mathsf{Var} \to \mathcal{U}$, the assignment *induced* by g is $G_g : \mathsf{Var}\times\mathsf{Var} \to \mathcal{U}\times\mathcal{U}$ such that $G_g(x, y) = (g(x), g(y))$, for each $x, y \in \mathsf{Var}$. The assignment $G : \mathsf{Var}\times\mathsf{Var} \to \mathcal{U}\times\mathcal{U}$ is *two-dimensional* when $G = G_{g_G}$.

Now, we will restrict HAL2 to two-dimensional assignments. This would make sense if two-dimensional assignments were closed under updates (appearing in the satisfiability condition for the binder). To have a two-dimensional assignment, we must propagate the update of (x, y) to other pairs of variables involving x or y. One way of achieving this propagation is by means of the induced assignment. The *coordinate update* of an assignment G is defined to be $(a/x)G := G_{(a/x)g_G}$. The *satisfiability* of a formula in a rooted square model $\mathcal{M} = \langle \mathcal{U} \times \mathcal{U}, C, R, I, V, (a, b)\rangle$ under a two-dimensional assignment $G : \mathsf{Var}\times\mathsf{Var} \to \mathcal{U}\times\mathcal{U}$ is defined as in AL, with the following extra clauses:

1. $\mathcal{M}, (a, b), G \models (x, y)$ iff $g_G(x) = a$ and $g_G(y) = b$.
2. $\mathcal{M}, (a, b), G \models @_{(x,y)}\alpha$ iff $\mathcal{M}, (g_G(x), g_G(y)), G \models \alpha$.
3. $\mathcal{M}, (a, b), G \models \downarrow_{(x,y)} \alpha$ iff $\mathcal{M}, (a, b), (a/x)(b/y)G \models \alpha$.

It is important to note that the binder operators in HAL and in HAL2 have distinct meanings, because of the difference in the definitions of updates of assignments.

We perform substitutions of variables. Following the semantical intuition, the substitutions are done coordinate by coordinate and these are made as in

FOL. The set of free (coordinate-)variables of a formula α, denoted by $\mathsf{free}(\alpha)$, and the notion of a (coordinate-)variable being *substitutable* for another in a given formula are as in FOL, considering that the down-arrow operator $\downarrow_{(x,y)}$ binds both x and y and that the satisfaction operator does not bind variables. The definition of substitution and substitutability for pair-variables come from the corresponding definitions for (coordinate-)variables. The result of the *substitution* of (z,w) for (x,y) in α, denoted by $((z,w)/(x,y))\alpha$, is $(w/y)(z/x)\alpha$. The pair-variable (z,w) is *substitutable* for (x,y) in α if w is substitutable for y in $(x/z)\alpha$ and z is substitutable for x in α.

In HAL2 the Substitution Lemma (for pair-variables) is established from a Substitution Lemma for coordinates.

Lemma 3.1 (Substitution for coordinates) *If y is substitutable for x in α, then $\mathcal{M},(a,b),(g_G(y)/x)G \models \alpha$ iff $\mathcal{M},(a,b),G \models (y/x)\alpha$.*

Corollary 3.2 (Substitution) *If (z,w) is substitutable for (x,y) in α, then $\mathcal{M},(a,b),(G(z,w)/(x,y))G \models \alpha$ iff $\mathcal{M},(a,b),G \models ((z,w)/(x,y))\alpha$.*

The axiom system for HAL2 is obtained from the axiom system for HAL by the following changes:
1) substitution of (capital-)variables for pair-variables;
2) exclusion of axiom AL;
3) restriction on axiom Q3, that will be stated as:
   Q3) $\downarrow_{(x,y)}((x,y) \to \alpha) \to \downarrow_{(x,y)} \alpha$, if $x \neq y$;
4) inclusion of the following axioms:
   Coordinates) $@_{(x,y)}(z,w) \leftrightarrow @_{(x,x)}(z,z) \wedge @_{(y,y)}(w,w)$.
   Comp) $(x,y) \to (x,z) \circ (z,y)$.
   Rev) $(x,y) \leftrightarrow \otimes(y,x)$.
   Id) $(x,x) \to \iota\delta$.

Coordinates expresses that the pair-variables (x,x) have a double role: to denote arrows in the identity relation and also coordinates of arrows. Id, Rev and Comp, together with Coordinates, force frames to be square frames and assignments to be two-dimensional. The restriction [2] in Q3 expresses syntactically the difference between the down-arrow operator semantics in HAL and in HAL2. As we state in Proposition 3.5, the axioms of AL are theorems of HAL2. Therefore, we can use theorems of HAL in the proof of results in HAL2, restricting theorems derived from Q3 to pair-variables (x,y) such that $x \neq y$.

Proposition 3.3 *The following are derived in* HAL2*:*
Bridge $\iota\delta$2) $@_{(x,x)}(y,y) \leftrightarrow @_{(x,y)}\iota\delta$.
Bridge $\otimes$ 2) $@_{(x,y)}\alpha \leftrightarrow @_{(y,x)} \otimes \alpha$.

[2] Q3 does not hold for pair-variables (x,x); for instance, when α is (x,x) we have that $\downarrow_{(x,x)}((x,x) \to (x,x))$ is valid but $\downarrow_{(x,x)}(x,x)$ is only satisfied in models rooted at arrows in the identity.

Bridge $\circ 2$) $@_{(x,z)}\alpha \wedge @_{(z,y)}\beta \to @_{(x,y)}(\alpha \circ \beta)$.

Paste $\circ 2$) $\dfrac{@_{(x,z)}\alpha \wedge @_{(z,y)}\beta \to \gamma}{@_{(x,y)}(\alpha \circ \beta) \to \gamma}$, *where z is new.*

Proof. By Name, Nom, Bridge $\otimes$, Bridge $\circ$, Coordinates, Id, Rev, Comp, Paste $\otimes$, Paste $\circ$, *Swap*, and **SEq**. □

Bridge $\iota\delta 2$, *Bridge* $\otimes 2$, *Bridge* $\circ 2$, and *Paste* $\circ 2$ reduce the Peircean operators to the hybrid logic operators, when in the scope of an satisfaction operator. They also provide an alternative axiomatization of HAL2: remove Bridge $\otimes$, Bridge $\circ$, Coordinates, Id, Rev, Comp, Paste $\otimes$, and Paste $\circ$, and add the two-dimensional versions of Bridge and Paste of Proposition 3.3 in place of them.

The following proposition is the key to the proofs of theorems in the non-hybrid arrow language, using the hybrid apparatus.

Proposition 3.4 T*) $\downarrow_{(x,y)} @_{(x,y)}\alpha \leftrightarrow \alpha$ *is a theorem of* HAL2, *if* $x \neq y$ *and* $x, y \notin \mathsf{free}(\alpha)$.

Proof. By Elimination, Q1, Q3, *Negation*, and **SEq**. □

The recipe is to take a new pair-variable (x, y) and to prove the theorem $@_{(x,y)}\alpha$. Then, using Normal, we obtain $\downarrow_{(x,y)} @_{(x,y)}\alpha$. And, finally, apply T* to obtain α. This recipe[3] is applied in the next proof, the pair-variables give the system the power to derive the missing arrow axioms. To prove this, we use the following results.

Proposition 3.5 *The* AL *axioms [19] are theorems of* HAL2.

Proof. It is not difficult. We exemplify with A5) $\alpha \circ \neg(\otimes\alpha \circ \beta) \to \neg\beta$:

Let $x, y \in Var$ s.t. $x \neq y$ and $x, y \notin \mathsf{free}(\alpha) \cup \mathsf{free}(\beta)$. Let $z \in Var$ s.t. z is distinct from x and y and z does not occur in α nor in β.

Bridge $\circ$, Negation	1	$@_{(z,x)} \otimes \alpha \to (@_{(z,y)}\neg(\otimes\alpha \circ \beta)) \to @_{(x,y)}\neg\beta)$
1, Bridge $\otimes$	2	$@_{(x,z)}\alpha \wedge @_{(z,y)}\neg(\otimes\alpha \circ \beta) \to @_{(x,y)}\neg\beta$
2, *Paste* $\circ 2$, Normal	3	$\downarrow_{(x,y)} @_{(x,y)}(\alpha \circ \neg(\otimes\alpha \circ \beta)) \to \downarrow_{(x,y)} @_{(x,y)}\neg\beta$
3, T*	4	$\alpha \circ \neg(\otimes\alpha \circ \beta) \to \neg\beta$

□

We can now prove the completeness of HAL2. All HAL axioms are axioms of HAL2 (axiom Q3 with the restriction $x \neq y$), with the exception of axiom AL, which is a theorem (Proposition 3.5). For HAL completeness, axiom Q3 is only used in the proof of Extended Lindenbaum Lemma (2.10) to show that the extension of a consistent set with a new variable is also consistent. In HAL2 we can extend a consistent set with a new pair-variable (x, y) with $x \neq y$ and use axiom Q3 with the restriction. Hence, to prove HAL2 completeness,

[3] In fact, we can use the preserving validity rule R*) from $@_{(x,y)}\alpha$ derive α, if $x \neq y$ and $x, y \notin \mathsf{free}(\alpha)$, instead of T*. Note that formula $@_{(x,y)}\alpha \to \alpha$ is not valid, however.

we shall consider the model $\mathcal{M}^\Gamma$ and the assignment G^Γ, provided by the completeness proof of HAL, and show that, when Γ is an MCS of HAL2, then the model $\mathcal{M}^\Gamma$ is a square and G^Γ is two-dimensional.

Lemma 3.6 *If Γ is an MCS in* HAL2*, then $\mathcal{M}^\Gamma$ is (isomorphic to) a square.*

Proof. We show that the function $h : S^\Gamma \to \{(\Delta_{(x,x)}, \Delta_{(y,y)}) : x, y \in \mathsf{Var}\}$ s.t. $h(\Delta^\Gamma_{(x,y)}) = (\Delta^\Gamma_{(x,x)}, \Delta^\Gamma_{(y,y)})$ is an arrow isomorphism. Using $\mathsf{Coordinates}$ and Lemma 2.5 we can prove that h is well defined and injective (h is surjective by construction); using *Bridge* $\iota\delta 2$, that h preserves identity; using $\mathsf{Coordinates}$, *Bridge* $\otimes 2$, and Lemma 2.6, that h preserves reversion; and using $\mathsf{Coordinates}$, *Bridge* $\circ 2$, *Paste* $\circ 2$, and Lemma 2.6, that h preserves composition. □

Lemma 3.7 *If Γ is an MCS in* HAL2*, then G^Γ is two-dimensional.*

Proof. The Satisfiability Lemma of HAL yields $\mathcal{M}^\Gamma, \Delta^\Gamma_{(x,y)}, G^\Gamma \models \mathsf{Comp} \wedge \mathsf{Id}$, whence we can show that $G^\Gamma = G_{g_{G^\Gamma}}$. □

Theorem 3.8 (Model Existence) *Every consistent set of* HAL2 *is satisfied in a rooted square model with a two-dimensional assignment.*

Proof. Given a consistent set Γ, extend Γ to a named MCS with witnesses Θ (as in HAL, using the Extended Lindenbaum Lemma with a pair-variable $X_0 = (x, y)$ with $x \neq y$). By Propositions 3.7 and 3.6, $\mathcal{F}^\Theta$ is a square and G^Θ is two-dimensional. Since Θ is named, take $(x, y) \in \Theta$. By the Satisfiability Lemma of HAL, $\mathcal{M}^\Theta, \Delta^\Theta_{(x,y)}, G^\Theta \models \alpha$, for each $\alpha \in \Gamma$. □

Finally, as a corollary we have the Completeness Theorem.

Theorem 3.9 (Completeness) *If $\Gamma \models_{\mathrm{HAL2}} \alpha$, then $\Gamma \vdash_{\mathrm{HAL2}} \alpha$.*

4 Relation Algebra with Binders

Relation Algebra with Binders (RAB) is presented in [15]. This is a hybridization of RA with variables denoting individual elements instead of pairs (arrows) and a one-dimensional version of the down-arrow operator $\downarrow$. In this section we describe a modal version of RAB called Arrow Logic with Binders (ALB), which is another hybridization of square arrow logic (AL2).

The *alphabet* of ALB is that of AL plus the *variables* $x_i : i \in \mathbb{N}$, and the binder $\downarrow^0$. The set of variables is denoted by Var, and x, y, z denote arbitrary variables. The *formulas* of ALB are obtained by closing the set of formulas of AL by the rules: $\alpha := x \mid \downarrow^0_x \alpha$. The square frames, square models and rooted square models for ALB are the same as for AL. Given a square model $\mathcal{M} = \langle \mathcal{U}\times\mathcal{U}, C, R, I, V \rangle$, an assignment to the individual variables is a function $g : \mathsf{Var} \to \mathcal{U}$. The *satisfiability* of a formula α at an arrow (a, b) of $\mathcal{M}$ under g is defined as in AL, with the following extra clauses:

1. $\mathcal{M}, (a, b), g \models x$ iff $g(x) = a = b$.

2. $\mathcal{M}, (a, b), g \models \downarrow^0_x \alpha$ iff $\mathcal{M}, (a, b), (a/x)g \models \alpha$.

In ALB, a binder $\downarrow^1$ for the other coordinate and two satisfaction operators, one for each coordinate, can be defined: $\downarrow^1_x \alpha := \otimes(\downarrow^0_x \alpha)$, $@^0_x\alpha := \top \circ x \circ \alpha$, and $@^1_x\alpha := \alpha \circ x \circ \top$.

The main result of [15] is that RAB has the expressive power of full FOL with the signature restricted to binary relations. In the modal formulation of the system, we have a similar result.

We now compare ALB with HAL2. We will show that both systems are equivalent with respect to expressive power and semantical consequence.

To compare ALB and HAL2 in means of expression and consequence we present an extension of each system. The first one, called ALB*, is an extension of ALB having HAL2 as a subsystem. The second one, called HAL2*, is an extension of HAL2 having ALB as a subsystem. Its easy to see that both ALB* and HAL2* have a common extension, showing that ALB and HAL2 are equipollent in means of expression and consequence [18].

ALB* is an extension by definition of ALB, obtained by adding pair-variables, satisfaction operators and binders of HAL2 to the language of ALB. The *alphabet* of ALB* is the alphabet of ALB plus the hybrid operators @ and $\downarrow$. The *formulas* of ALB* are obtained by closing the set of ALB formulas by the rules: $\alpha := (x, y) \mid @_{(x,y)}\alpha \mid \ \downarrow_{(x,y)} \alpha$. Hence, the language of HAL2 is a sublanguage of the language of ALB*. The *satisfiability* conditions for a ALB*-formula in a rooted square model $\mathcal{M} = \langle \mathcal{U} \times \mathcal{U}, C, R, I, V, (a, b)\rangle$ under assignment g are the same as for ALB with the following extra clauses:

1. $\mathcal{M}, (a, b), g \models (x, y)$ iff $\mathcal{M}, (a, b), g \models x \circ \top \circ y$.
2. $\mathcal{M}, (a, b), g \models @_{(x,y)}\alpha$ iff $\mathcal{M}, (a, b), g \models @^0_x @^1_y \alpha$.
3. $\mathcal{M}, (a, b), g \models \downarrow_{(x,y)} \alpha$ iff $\mathcal{M}, (a, b), g \models \downarrow^0_x \downarrow^1_y \alpha$.

We prove that HAL2 is a subsystem of ALB*, by using the following lemma, whose proof is by a straightforward induction on formulas.

Lemma 4.1 *For all square model* $\mathcal{M} = \langle \mathcal{U} \times \mathcal{U}, C, R, I, V\rangle$, HAL2-*formula* α, ALB-*assignment* g, *and* $a, b \in \mathcal{U}$ *we have:*

$$\mathcal{M}, (a, b), g \models_{\text{ALB}^*} \alpha \text{ iff } \mathcal{M}, (a, b), G_g \models_{\text{HAL2}} \alpha.$$

Theorem 4.2 *If* $\Gamma \cup \{\alpha\}$ *is a set of* HAL2-*formulas, then* $\Gamma \models_{\text{HAL2}} \alpha$ *implies* $\Gamma \models_{\text{ALB}^*} \alpha$.

Proof. Assume $\mathcal{M}, (a, b), g \models_{\text{ALB}^*} \Gamma$. Then $\mathcal{M}, (a, b), G_g \models_{\text{HAL2}} \Gamma$, by Lemma 4.1. Since $\Gamma \models_{\text{HAL2}} \alpha$, we have that $\mathcal{M}, (a, b), G_g \models_{\text{HAL2}} \alpha$. By Lemma 4.1 again, $\mathcal{M}, (a, b), g \models_{\text{ALB}^*} \alpha$. □

HAL2* is an extension by definition of HAL2, obtained by adding one-dimensional variables and the binders of ALB to the language of HAL2.

The *alphabet* of HAL2* is the alphabet of HAL2 together with $\downarrow^0$. The *formulas* of HAL2* are obtained by closing the set of formulas HAL2 by the rules: $\alpha := x \mid \ \downarrow^0_x \alpha$. Hence, the language of ALB is a sublanguage of the language of HAL2*. The *satisfiability* conditions for the HAL2*-formulas in a rooted square model $\mathcal{M} = \langle \mathcal{U} \times \mathcal{U}, C, R, I, V, (a, b)\rangle$ under assignment G are the same as for HAL2 with the following extra clauses:

1. $\mathcal{M}, (a, b), G \models x$ iff $\mathcal{M}, (a, b), G \models (x, x)$.
2. $\mathcal{M}, (a, b), G \models \downarrow_x^0 \alpha$ $\mathcal{M}, (a, b), G \models \downarrow_{(x,y)} \alpha$, where y does not occur in α.

We prove that ALB is a subsystem of HAL2*, by using the following lemma, whose proof is by a straightforward induction on formulas.

Lemma 4.3 *For all square model $\mathcal{M} = \langle \mathcal{U} \times \mathcal{U}, C, R, I, V\rangle$,* ALB*-formula α,* HAL2*-assignment G, and $a, b \in \mathcal{U}$:*

$$\mathcal{M}, (a, b), G \models_{\mathrm{HAL2}^*} \alpha \textit{ iff } \mathcal{M}, (a, b), g_G \models_{\mathrm{ALB}} \alpha.$$

Theorem 4.4 *If $\Gamma \cup \{\alpha\}$ is a set of* ALB*-formulas, $\Gamma \models_{\mathrm{ALB}} \alpha$ implies $\Gamma \models_{\mathrm{HAL2}^*} \alpha$.*

Proof. Assume $\mathcal{M}, (a, b), G \models_{\mathrm{HAL2}^*} \Gamma$. Then $\mathcal{M}, (a, b), g_G \models_{\mathrm{ALB}} \Gamma$, by Lemma 4.3. Since $\Gamma \models_{\mathrm{ALB}} \alpha$, we have $\mathcal{M}, (a, b), g_G \models_{\mathrm{ALB}} \alpha$. By Lemma 4.3 again, $\mathcal{M}, (a, b), G \models_{\mathrm{HAL2}^*} \alpha$. □

In [10], we have an axiomatization of RAB. An alternative axiomatization is obtained from the comparison of ALB and HAL2, using the definitions of the HAL2 operators in ALB.

5 Perspective

In this paper we investigate two approaches to hybridizing Arrow Logic (AL): one, HAL, under abstract semantics and another, HAL2, under square semantics.

HAL2 is a system obtained by a direct application of the standard process of hybridization to Arrow Logic under Square Semantics (AL2). To make the hybridization of AL2 smoother, we have proceeded in two steps. First, we have obtained HAL from AL under abstract semantics, following directly the standards of hybridization. (In particular, we have presented an axiomatization of HAL by extending the results in [8].) Second, we have defined HAL2 by making two essential modifications in HAL: introducing pair-variables and restricting the semantics to square frames and satisfaction to the subclass of assignments that preserve the two-dimensional structure of the variables. We have presented an axiomatization of HAL2 and have proven that this system is equivalent to Relation Algebra with Binders (RAB). As a consequence of our results we have obtained a modular axiomatization of RAB (in [10] an alternative axiomatization was presented).

The main feature of HAL2 is the use of pair-variables to denote pair of points of the square frames. The adequacy of this approach to the symbolization of elementary statements about binary relations was illustrated in Section 1. A systematic investigation of the use of pair-variables seems to be a very interesting topic of research.

In [20] an axiomatization that is strongly sound and complete w.r.t. square frames was presented by introducing a version of the irreflexivity rule [12] for the difference operator [11]. This rule makes the system somewhat inadequate for proving theorems. The axiomatization presented here, besides being modular, appears to be much easier to use, since its axioms and rules are handled in a natural manner. The study of the proof theory of HAL2 and its relation with the proof of theorems in Venema's system seems to be a natural development to be pursued.

The natural hybridization we propose has the following characteristics: 1) introduction of *variables* to denote arrows; 2) introduction of *special operators* (like @ and ↓) for handling information now available due to the new variables. Such natural hybridizations can be expected to provide standard ways of extending the expressive power of Arrow Logic under other semantics.

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