Natural Deduction for First-Order Hybrid Logic

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Abstract

This is a companion paper to [6] where a natural deduction system for propositional hybrid logic is given. In the present paper we generalize the system to the first-order case. Our natural deduction system for first-order hybrid logic can be extended with additional inference rules corresponding to conditions on the accessibility relations and the quantifier domains expressed by so-called geometric theories. We prove soundness and completeness and we prove a normalisation theorem.

1 Introduction

In this paper the natural deduction system for propositional hybrid logic given in [6] is generalized to the first-order case. Our first-order machinery includes first-order variables, so-called grounded definite descriptions, predicates, equality, and of course first-order quantifiers.

Now, propositional hybrid logic is obtained by adding to ordinary propositional multi-modal logic further expressive power in the form of so-called satisfaction operators and a second sort of propositional symbols called nominals. A nominal is assumed to be true at exactly one world, so a nominal can be considered the name of a world. Thus, in hybrid logic a name is a particular sort of propositional letter whereas in first-order logic it is an argument to a predicate. Beside nominals, a satisfaction operator $a :$ is added for each nominal a. A satisfaction operator $a :$ makes it possible to express that a formula is true at one particular world, namely the world at which the nominal a is true. Moreover, we shall consider the so-called binders $\forall$ and $\downarrow$. The binders bind nominals to worlds in two different ways: The $\forall$ binder quantifies over worlds whereas $\downarrow$ binds a nominal to the actual world. The $\downarrow$ binder is definable in terms of $\forall$. First-order hybrid logic is obtained by generalizing propositional hybrid logic to cover first-order machinery, cf. above.

The history of hybrid logic goes back to Arthur Prior's work, see [12], Chapter XI. See [9] for an account of Prior's work. The paper [5] makes a connection to Donald Davidson's notion of a theory of truth.

Natural deduction style inference rules for ordinary classical first-order logic were originally introduced by Gerhard Gentzen and later on considered by Dag Prawitz in [10] and [11]. In the present paper we shall give natural deduction systems for various fragments of first-order hybrid logic and we shall prove that the systems are sound and complete. Moreover, we shall show how to extend the natural deduction systems with additional inference rules corresponding to first-order conditions on the accessibility relations and the quantifier domains. The conditions we consider are expressed by so-called geometric theories. Different geometric theories give rise to different hybrid logics, so natural deduction systems for new hybrid logics can be obtained in a uniform way simply by adding inference rules as appropriate. First-order conditions expressed by geometric theories covers a very wide class of logics. This is for example witnessed by the fact that any so-called Geach axiom schema, that is, modal logical axiom schema of the form $\Diamond^k\Box^m\phi \rightarrow \Box^l\Diamond^n\phi$ corresponds to a formula of the form required in a geometric theory (of course, $\Box^j$ is an abbreviation for a sequence of j occurrences of $\Box$, etc.).

Natural deduction systems are characterised by having two different kinds of rules for each non-nullary connective; there is a kind of rules which introduces a connective and there is a kind of rules which eliminates a connective. A maximum formula in a derivation is then a formula occurrence that is both introduced by an introduction rule and eliminated by an elimination rule. Such a maximum formula can be considered a "detour" in the derivation. Maximum formulas can be removed by using what is called proper reductions. Another kind of reductions, permutative reductions, are used to remove so-called permutative formulas. A derivation is called normal if it contains no maximum or permutative formula and we give a normalisation theorem which says that any derivation can be rewritten to such a normal derivation by repeated applications of reductions.

This paper is structured as follows. In the second section of the paper we introduce the basics of first-order hybrid logic, in the third section we introduce our natural deduction systems, and in the fourth section we prove soundness and completeness. The fifth section is concerned with normalisation.

2 First-Order Hybrid Logic

In this section we introduce the basics of first-order hybrid logic. We shall in many cases adopt the terminology of [4]. See also [7] and [8]. The first-order hybrid logic we consider is obtained by adding a sort of propositional symbols called *nominals* to first-order multi-modal logic with identity, that is, first-order logic with identity extended with a finite number of modal operators $\Box_1, \ldots, \Box_m$. (So nominals are the only sort of propositional symbols, but 0-

place predicate symbols of course corresponds to propositional symbols in the ordinary sense.) It is assumed that a countably infinite set of nominals and a countably infinite set of ordinary first-order variables are given. The sets are assumed to be disjoint. The metavariables a, b, ... range over nominals and x, y, ... range over first-order variables.

Beside nominals, two so-called *binders*, $\forall$ and $\downarrow$, and for each nominal a, an operator $a :$ called a *satisfaction operator* are added. We furthermore assume that a set of so-called *non-rigid designators*, disjoint from the set of nominals, is given, and we follow [4] in overloading the notation for the satisfaction operator by defining a term to be either a first-order variable or an expression of the form $a : q$ where a is a nominal and q is a non-rigid designator. Of course, the term $a : q$ denotes the value of q at the world where a is true. Such terms are called *grounded definite descriptions*. The formulas of first-order hybrid logic are defined by the grammar

$$S ::= P(t_1, \ldots, t_n) \,|\, t = u \,|\, a \,|\, S \wedge S \,|\, S \rightarrow S \,|\, \bot \,|\, \Box_i S \,|\, a : S \,|\, \forall x S \,|\, \forall a S \,|\, {\downarrow} a S$$

where P is a n-place predicate symbol, t_1, ..., t_n as well as t and u are terms, a is a nominal, and x is an ordinary first-order variable. In what follows, the metavariables ϕ, ψ, ... range over formulae. Negation, nullary conjunction, and disjunction are defined by the conventions that $\neg\phi$ is an abbreviation for $\phi \rightarrow \bot$, $\top$ is an abbreviation for $\neg\bot$, and $\phi \vee \psi$ is an abbreviation for $\neg(\neg\phi \wedge \neg\psi)$. Similarly, $\Diamond_i\phi$ is an abbreviation for $\neg\Box_i\neg\phi$ and $\exists x\phi$ is an abbreviation for $\neg\forall x\neg\phi$ (and analogously when the first-order variable x is replaced by a nominal a). Moreover, the so-called *existence predicate* is defined by the convention that $E(t)$ is an abbreviation for $\exists y(y = t)$ where y is a variable distinct from any variable occuring in t. It is assumed that we are working with a fixed number of modalities.

We now define skeletons and models. Skeletons are first-order versions of the usual frames for propositional multi-modal logic. A *skeleton* is a tuple $(W, R_1, \ldots, R_m, D, \delta)$ where W is a non-empty set, R_1, ..., R_m are 2-place relations on W, D is a non-empty set, and δ is a function that to each element w of W assigns a subset of D. The elements of W are called *worlds*, each relation R_i is called an *accessibility relation*, and $\delta(w)$ is called the *domain of quantification* at the world w. A *model* is a tuple $(W, R_1, \ldots, R_m, D, \delta, V)$ where $(W, R_1, \ldots, R_m, D, \delta)$ is a skeleton and V is a *valuation*, that is, V is a function that to each pair consisting of a world w and a non-rigid designator assigns an element of D, and moreover, to each pair consisting of a world w and an n-place predicate symbol assigns a subset of D^n. The first-order model $(W, R_1, \ldots, R_m, D, \delta, V)$ is *based* on the skeleton $(W, R_1, \ldots, R_m, D, \delta)$.

An *assignment* for a model $\mathcal{M} = (W, R_1, \ldots, R_m, D, \delta, V)$ is a function that to each nominal assigns an element of W and to each first-order variable assigns an element of D. Given an assignment g, each term t is assigned an element $t^{\mathcal{M},g}$ of D as follows: If t is of the form $a : q$, then $t^{\mathcal{M},g} = V(g(a), q)$, otherwise t is a variable, in which case $t^{\mathcal{M},g} = g(t)$. Given assignments g' and

g, $g' \stackrel{x}{\sim} g$ means that g' agrees with g on all nominals and first-order variables save possibly on the first-order variable x (and analogously if x is replaced by a nominal a). Given a model $\mathcal{M}$, the relation $\mathcal{M}, w \models \phi[g]$ is defined by induction, where w is a world, g is an assignment, and ϕ is a formula of first-order hybrid logic.

$$\begin{aligned}
\mathcal{M}, w \models P(t_1, \ldots, t_n)[g] & \text{ iff } (t_1^{\mathcal{M},g}, \ldots, t_n^{\mathcal{M},g}) \in V(w, P) \\
\mathcal{M}, w \models t = u[g] & \text{ iff } t^{\mathcal{M},g} = u^{\mathcal{M},g} \\
\mathcal{M}, w \models a[g] & \text{ iff } w = g(a) \\
\mathcal{M}, w \models \Box_i \phi[g] & \text{ iff for any } v \in W \text{ where } wR_iv,\ \mathcal{M}, v \models \phi[g] \\
\mathcal{M}, w \models a : \phi[g] & \text{ iff } \mathcal{M}, g(a) \models \phi[g] \\
\mathcal{M}, w \models \forall x \phi[g] & \text{ iff for any } g' \stackrel{x}{\sim} g \text{ where } g'(x) \in \delta(w),\ \mathcal{M}, w \models \phi[g'] \\
\mathcal{M}, w \models \forall a \phi[g] & \text{ iff for any } g' \stackrel{a}{\sim} g,\ \mathcal{M}, w \models \phi[g'] \\
\mathcal{M}, w \models \downarrow a \phi[g] & \text{ iff } \mathcal{M}, w \models \phi[g'] \text{ where } g' \stackrel{a}{\sim} g \text{ and } g'(a) = w
\end{aligned}$$

The connectives $\wedge$, $\rightarrow$, and $\bot$ are interpreted as usual. A formula ϕ is said to be *true* at the world w if $\mathcal{M}, w \models \phi[g]$; otherwise it is said to be *false* at w. By convention $\mathcal{M} \models \phi[g]$ means $\mathcal{M}, w \models \phi[g]$ for every world w and $\mathcal{M} \models \phi$ means $\mathcal{M} \models \phi[g]$ for every assignment g. A formula ϕ is *valid* in a skeleton if and only if $\mathcal{M} \models \phi$ for any model $\mathcal{M}$ that is based on the skeleton in question. A formula ϕ is *valid* in a class of skeletons $\mathbf{F}$ if and only if ϕ is valid in any skeleton in $\mathbf{F}$.

Note that we use the same notation for the binder $\forall$ and for the first-order quantifier. Also, note that we allow the domain of a first-order quantifier to vary from world to world and we allow a domain to be empty. Moreover, we allow a non-rigid designator to designate a non-existent and we allow a predicate to be true of non-existents.

The notions of free and bound occurrences of nominals and first-order variables are defined in the obvious way. Also, if $\overline{x}$ is a list of pairwise distinct first-order variables and $\overline{t}$ is a list of terms of the same length as $\overline{x}$, then $\psi[\overline{t}/\overline{x}]$ is the formula ψ where the terms $\overline{t}$ have been simultaneously substituted for all free occurrences of the variables $\overline{x}$. It is assumed that no variable x_i in $\overline{x}$ occur free in ψ within the scope of $\forall y$ where y is any first-order variable occuring in t_i or within the scope of $\forall a$ or $\downarrow a$ where a is any nominal occuring in t_i. An analogous definition is obtained if the lists $\overline{x}$ and $\overline{t}$ are replaced by lists of nominals as appropriate. Now, let $\mathcal{O} \subseteq \{\downarrow, \forall\}$. In what follows $\mathcal{H}(\mathcal{O})$ denotes the fragment of first-order hybrid logic in which the only binders are the binders in the set $\mathcal{O}$. It is assumed that the set $\mathcal{O}$ is fixed unless other is specified.

It is straightforward that a model for first-order hybrid logic can be considered a model for two-sorted first-order logic and vice versa. Obviously, there is one sort for worlds and one for individuals. Terms of the two-sorted first-order language in question are built out of variables ranging over worlds, variables ranging over individuals, and for each non-rigid designator, a unary function symbol which is interpreted as a function from worlds to individuals. Thus, all terms ranging over worlds are variables and a term ranging over individuals is either a variable or of the form $q(a)$ where q is a non-rigid designator and a is a variable ranging over worlds. Formulas of the two-sorted first-order language are defined by the grammar

$$S ::= P(a, t_1, \ldots, t_n) | R_i(a, b) | E(a, t) | a = b | t = u | S \wedge S | S \rightarrow S | \bot | \forall a S | \forall x S$$

where P is a n-place predicate symbol of first-order modal logic, a and b are variables ranging over worlds, and $t_1, \ldots, t_n$ as well as t and u are terms ranging over individuals. Note that the language contains two equality predicates and two quantifiers; an equality predicate and a quantifier for each sort. The two-sorted $(n+1)$-place predicate symbol P is interpreted such that it relativizes the interpretation of the corresponding modal n-place predicate symbol to worlds, the predicate symbol R_i is interpreted using the appropriate accessibility relation, and the predicate symbol E is interpreted such that it relates a world to individuals existing at that world. The connectives $\neg$, $\top$, $\vee$, and $\exists$ are defined in one of the usual ways. Free and bound variables are defined in the obvious way.

3 Natural Deduction for First-Order Hybrid Logic

In this section we give natural deduction inference rules for the first-order hybrid logic $\mathcal{H}(\mathcal{O})$. Moreover, we show how to extend the system with additional rules corresponding to conditions on the accessibility relations and the quantifier domains.

Natural deduction systems are characterised by having two different kinds of rules for each non-nullary connective; there is a kind of rules which introduces a connective, such rules are called *introduction rules*, and similarly, there is a kind of rules which eliminates a connective, such rules are called *elimination rules*. Natural deduction inference rules for first-order hybrid logic are given in Figure 1, Figure 2, and Figure 3. The rules given are the natural deduction rules for propositional hybrid logic given in [6] together with rules for first-order quantification, that is, $(\forall I1)$ and $(\forall E1)$, rules for first-order equality, namely $(\mathit{Ref}2)$ and $(\mathit{Rep}1)$, and moreover, rules for equality in connection with existence predicates and non-rigid designators, namely $(\mathit{Nom}3)$ and $(\mathit{Nom}4)$. The rule $(\mathit{Nom}4)$ is literally the same as a tableau rule given in [4]. It is instructive to compare the rules for first-order equality with the rules for equality in connection with nominals $(\mathit{Ref}1)$, $(\mathit{Nom}1)$, $(\mathit{Nom}2)$, and $(\mathit{Nom}3)$. A natural question to ask is why $(\mathit{Rep}1)$ and $(\mathit{Nom}1)$ are restricted to atomic formulas (which furthermore have to be different from $\bot$). The reason is that the system obtained by removing these restrictions does not enjoy an appropriate version of the subformula property. The reason why $(\bot 1)$ is re-

$$\frac{a : \phi \quad a : \psi}{a : (\phi \wedge \psi)} (\wedge I) \qquad \frac{a : (\phi \wedge \psi)}{a : \phi} (\wedge E1) \qquad \frac{a : (\phi \wedge \psi)}{a : \psi} (\wedge E2)$$

$$\begin{array}{c} [a : \phi] \\ \vdots \\ a : \psi \end{array}$$
$$\frac{a : \psi}{a : (\phi \to \psi)} (\to I) \qquad \frac{a : (\phi \to \psi) \quad a : \phi}{a : \psi} (\to E)$$

$$\begin{array}{c} [a : E(z)] \\ \vdots \\ a : \phi[z/x] \end{array}$$
$$\frac{a : \phi[z/x]}{a : \forall x \phi} (\forall I1)^{*} \qquad \frac{a : \forall x \phi \quad a : E(t)}{a : \phi[t/x]} (\forall E1)$$

$$\begin{array}{c} [a : \neg\phi] \\ \vdots \\ a : \bot \end{array}$$
$$\frac{a : \bot}{a : \phi} (\bot 1)^{\star} \qquad \frac{a : \bot}{c : \bot} (\bot 2)$$

∗ z does not occur free in $a : \forall x \phi$ or in any undischarged assumptions other than the specified occurrences of $a : E(z)$.

⋆ ϕ is an atomic formula different from $\bot$.

Fig. 1. Natural deduction rules: Propositional and first-order connectives

stricted is similar (and well-known from the literature, cf. [10] and [11]). The unrestricted versions of the rules are however admissible, cf. Proposition 4.1.

All formulas in the rules are of the form $a : \phi$. All rules with a name on the form $(\ldots I \ldots)$ are introduction rules, and similarly, all rules with a name on the form $(\ldots E \ldots)$ are elimination rules. Note that the rules $(\bot 1)$ and $(\bot 2)$ are neither introduction rules nor elimination rules (recall that $\neg\phi$ is an abbreviation for $\phi \to \bot$). A derivation is generated from the inference rules from derivations consisting of a single undischarged assumption. The rule instance at which a discharged assumption is discharged is indicated by marking the assumption and the rule instance with identical numbers. We shall frequently omit this information when no confusion can occur.

We shall make use of the following conventions: The metavariables π, τ, ... range over derivations. We shall call formulas of the form $a : \phi$ *satisfaction statements*, cf. a similar notion in [3]. The metavariables Γ, Δ, ... range over sets of satisfaction statements. A derivation π is a *derivation of* $a : \phi$ *from* Γ if and only if the end-formula of π is an occurrence of $a : \phi$ and each undischarged assumption in π is an occurrence of a formula in Γ. Moreover, $\pi[\overline{t}/\overline{x}]$ is the derivation π where each formula occurrence ψ has been replaced by $\psi[\overline{t}/\overline{x}]$. An analogous definition is obtained if the first-order variables $\overline{x}$ and the terms $\overline{t}$ are replaced by nominals as appropriate.

$$\dfrac{\begin{array}{c}[a : \Diamond_i c]\\ \vdots \\ c : \phi\end{array}}{a : \Box_i \phi}\ (\Box_i I)^*
\qquad
\dfrac{a : \Box_i \phi \quad a : \Diamond_i e}{e : \phi}\ (\Box_i E)$$

$$\dfrac{a : \phi}{c : a : \phi}\ (: I)
\qquad
\dfrac{c : a : \phi}{a : \phi}\ (: E)$$

$$\dfrac{\begin{array}{c}[a : c]\\ \vdots \\ c : \phi[c/b]\end{array}}{a :\downarrow b\phi}\ (\downarrow I)^\star
\qquad
\dfrac{a :\downarrow b\phi \quad a : e}{e : \phi[e/b]}\ (\downarrow E)$$

$$\dfrac{a : \phi[c/b]}{a : \forall b\phi}\ (\forall I2)^\dagger
\qquad
\dfrac{a : \forall b\phi}{a : \phi[e/b]}\ (\forall E2)$$

$*$ c does not occur free in $a : \Box_i\phi$ or in any undischarged assumptions other than the specified occurrences of $a : \Diamond_i c$.

$\star$ c does not occur free in $a :\downarrow b\phi$ or in any undischarged assumptions other than the specified occurrences of $a : c$.

$\dagger$ c does not occur free in $a : \forall b\phi$ or in any undischarged assumptions.

Fig. 2. Natural deduction rules: Modal and hybrid connectives

$$\dfrac{}{a : a}\ (Ref1)
\quad
\dfrac{a : c \quad a : \phi}{c : \phi}\ (Nom1)^*
\quad
\dfrac{a : c \quad a : \Diamond_i b}{c : \Diamond_i b}\ (Nom2)
\quad
\dfrac{a : c \quad a : E(t)}{c : E(t)}\ (Nom3)$$

$$\dfrac{}{a : (t = t)}\ (Ref2)
\quad
\dfrac{a : (t = u) \quad c : \phi[t/x]}{c : \phi[u/x]}\ (Rep1)^*
\quad
\dfrac{a : c}{b : ((a : q) = (c : q))}\ (Nom4)$$

$*$ ϕ is an atomic formula different from $\bot$.

Fig. 3. Natural deduction rules: Nominals and first-order terms

Our natural deduction system for $\mathcal{H}(\mathcal{O})$ is obtained from the rules given in Figure 1, Figure 2, and Figure 3 by leaving out the rules for the binders that are not in the set $\mathcal{O}$ (recall that $\mathcal{O} \subseteq \{\downarrow, \forall\}$). The system thus obtained will be denoted $\mathbf{N}_{\mathcal{H}(\mathcal{O})}$. So for example $\mathbf{N}_{\mathcal{H}(\{\forall\})}$ is obtained by leaving out the rules $(\downarrow I)$ and $(\downarrow E)$. The natural deduction system $\mathbf{N}_{\mathcal{H}(\mathcal{O})}$ corresponds to the class of all skeletons, that is, the class of skeletons where no conditions are imposed on the accessibility conditions or the quantifier domains. Hence, it is a first-order and hybrid version of the standard modal logic K.

$$\dfrac{s_1[\overline{t}/\overline{x}] \quad \ldots \quad s_n[\overline{t}/\overline{x}] \qquad \begin{array}{c}[s_{11}[\overline{t}/\overline{x}]] \ldots [s_{1n_1}[\overline{t}/\overline{x}]] \\ \vdots \\ \phi\end{array} \qquad \ldots \qquad \begin{array}{c}[s_{m1}[\overline{t}/\overline{x}]] \ldots [s_{mn_m}[\overline{t}/\overline{x}]] \\ \vdots \\ \phi\end{array}}{\phi} \; (R_\theta)^*$$

$*$ $\overline{t}$ is any list of terms of the same length as $\overline{x}$, none of the nominals in $\overline{c}$ and the first-order variables in $\overline{y}$ occur in any terms in $\overline{t}$, and none of the nominals in $\overline{c}$ and the first-order variables in $\overline{y}$ occur free in ϕ or in any undischarged assumptions other than the specified occurrences of s_{jk}.

Fig. 4. Natural deduction rules: Geometric theories

3.1 Conditions on the accessibility relations

In what follows we shall consider natural deduction systems obtained by extending $\mathbf{N}_{\mathcal{H}(\mathcal{O})}$ with additional inference rules corresponding to first-order conditions on the accessibility relations and the quantifier domains. The conditions we consider are expressed by so-called geometric theories. A two-sorted first-order formula is *geometric* if it is built out of atomic formulas of the forms $R_i(a, c)$, $E(a, x)$, $a = c$, and $x = y$ using only the connectives $\bot$, $\wedge$, $\vee$, and $\exists$. In what follows, the metavariables S_k and S_{jk} range over atomic formulas of the mentioned forms. Atomic formulas of the mentioned forms can be translated into first-order hybrid logic in a truth preserving way as follows.

$$HT(R_i(a, c)) = a : \Diamond_i c \qquad HT(a = c) = a : c$$
$$HT(E(a, x)) = a : E(x) \qquad HT(x = y) = b : (x = y)$$

The nominal b in the last clause is arbitrary (we want $HT(x = y)$ to be a satisfaction statement, cf. the definition of the natural deduction rule (R_θ) below, so we simply prefix $x = y$ by an arbitrary satisfaction operator). See [14] for an introduction to geometric logic.

Now, a *geometric theory* is a finite set of closed two-sorted first-order formulas each having the form $\forall \overline{a}\, \overline{x}(\phi \rightarrow \psi)$ where the formulas ϕ and ψ are geometric, $\overline{a}$ is a list $a_1, \ldots, a_l$ of variables ranging over worlds, $\overline{x}$ is a list $x_1, \ldots, x_h$ of variables ranging over individuals, and $\forall \overline{a}\, \overline{x}$ is an abbreviation for $\forall a_1 \ldots \forall a_l \forall x_1 \ldots \forall x_h$. It can be shown that any geometric theory is equivalent to a *basic geometric theory* which is a geometric theory in which each formula has the form

$$\forall \overline{a}\, \overline{x}((S_1 \wedge \ldots \wedge S_n) \rightarrow \exists \overline{c}\, \overline{y} \bigvee_{j=1}^{m} (S_{j1} \wedge \ldots \wedge S_{jn_j}))$$

where $n, m \geq 0$ and $n_1, \ldots, n_m \geq 1$. For simplicity, we assume that the variables in each of the lists $\overline{a}$ and $\overline{x}$ are pairwise distinct, that the variables in each of $\overline{c}$ and $\overline{y}$ are pairwise distinct, and that no variable occurs in both $\overline{a}$ and $\overline{c}$ or in both $\overline{x}$ and $\overline{y}$. Note that a formula of the form displayed above is a Horn clause if $\overline{c}$ and $\overline{y}$ are empty, $m = 1$, and $n_m = 1$.

We now give hybrid natural deduction rules corresponding to a basic geometric theory. The metavariables s_k and s_{jk} range over hybrid formulas of the forms $a : \Diamond_i c$, $a : E(x)$, $a : c$, and $b : (x = y)$. With a two-sorted first-order formula θ of the form displayed above, we associate the natural deduction inference rule (R_θ) given in Figure 4 where s_k is of the form $HT(S_k)$ and s_{jk} is of the form $HT(S_{jk})$. For example, if θ is the formula

$$\forall a \forall c \forall x ((R_i(a,c) \wedge E(a,x)) \rightarrow E(c,x))$$

then (R_θ) is the rule

$$\dfrac{a : \Diamond_i c \qquad a : E(t) \qquad \begin{array}{c}[c : E(t)] \\ \vdots \\ \phi\end{array}}{\phi} \, (R_\theta)$$

The formula, and hence the inference rule, corresponds to the quantifier domains being increasing wrt. the accessibility relation R_i. Now, let $\mathbf{T}$ be any basic geometric theory. The natural deduction system obtained by extending $\mathbf{N}_{\mathcal{H}(\mathcal{O})}$ with the set of rules $\{(R_\theta) \mid \theta \in \mathbf{T}\}$ will be denoted $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$. We shall assume that we are working with a fixed basic geometric theory $\mathbf{T}$ unless other is specified.

It is straightforward to check that if a formula in a basic geometric theory is a Horn clause, then the rule (R_θ) given in Figure 4 can be replaced by the following simpler rule (which we have also called (R_θ)).

$$\dfrac{s_1[\overline{t}/\overline{x}] \quad \ldots \quad s_n[\overline{t}/\overline{x}]}{s_{11}[\overline{t}/\overline{x}]} \, (R_\theta)$$

In the rule, $\overline{t}$ is any list of terms of the same length as $\overline{x}$. Note, by the way, that the rules (*Ref*1), (*Ref*2), (*Nom*2), and (*Nom*3) are all of this form. For example, if θ is the formula corresponding to increasing domains, cf. above, then the following rule will do.

$$\dfrac{a : \Diamond_i c \qquad a : E(t)}{c : E(t)} \, (R_\theta)$$

Natural deduction rules corresponding to Horn clauses were discussed already in [11].

Below is an example of a derivation in $\mathbf{N}_{\mathcal{H}(\emptyset)} + \{\theta\}$ where θ is the formula corresponding to increasing domains and where we have used the simplified version of (R_θ).

$$\dfrac{\dfrac{[a:\Box_i\forall x\phi]^3 \quad [a:\Diamond_i c]^1}{c:\forall x\phi}(\Box_i E) \qquad \dfrac{[a:\Diamond_i c]^1 \quad [a:E(x)]^2}{c:E(x)}(R_\theta)}{\dfrac{\dfrac{\dfrac{c:\phi}{a:\Box_i\phi}(\Box_i I)^1}{a:\forall x\Box_i\phi}(\forall I1)^2}{a:(\Box_i\forall x\phi\rightarrow\forall x\Box_i\phi)}(\rightarrow I)^3}(\forall E1)$$

The nominal c is new. The end-formula of the derivation is the well-known *Converse Barcan Formula* prefixed by a satisfaction operator.

4 Soundness and Completeness

The aim of this section is to prove soundness and completeness. The proof is an extension of the proof given in [6] in connection with propositional hybrid logic. We shall therefore skip the parts of the proof covered by [6]. Also, further references can be found in that paper. In what follows we shall need the convention that the *degree* of a formula is the number of occurrences of non-nullary connectives in it.

Proposition 4.1 *The rules*

$$\begin{array}{c}[a:\neg\phi]\\ \vdots\\ \dfrac{c:\bot}{a:\phi}(\bot)\end{array} \qquad \dfrac{a:c \quad a:\phi}{c:\phi}(Nom) \qquad \dfrac{a:(t=u) \quad c:\phi[t/x]}{c:\phi[u/x]}(Rep)$$

are admissible in $\mathbf{N}_{\mathcal{H}(\mathcal{O})}+\mathbf{T}$.

Proof. The proof makes use of the notion of degree, cf. above, and it is analogous to a proof given in [6]. □

Note in the proposition above that ϕ can be any formula.

Definition 4.2 A set of satisfaction statements Γ in $\mathcal{H}(\mathcal{O})$ is $\mathbf{N}_{\mathcal{H}(\mathcal{O})}+\mathbf{T}$-*inconsistent* if and only if $a:\bot$ is derivable from Γ in $\mathbf{N}_{\mathcal{H}(\mathcal{O})}+\mathbf{T}$ for some nominal a and Γ is $\mathbf{N}_{\mathcal{H}(\mathcal{O})}+\mathbf{T}$-*consistent* if and only if Γ is not $\mathbf{N}_{\mathcal{H}(\mathcal{O})}+\mathbf{T}$-inconsistent. Moreover, Γ is *maximal* $\mathbf{N}_{\mathcal{H}(\mathcal{O})}+\mathbf{T}$-*consistent* if and only if Γ is $\mathbf{N}_{\mathcal{H}(\mathcal{O})}+\mathbf{T}$-consistent and any set of satisfaction statements in $\mathcal{H}(\mathcal{O})$ that properly extends Γ is $\mathbf{N}_{\mathcal{H}(\mathcal{O})}+\mathbf{T}$-inconsistent.

We shall frequently omit the reference to $\mathcal{H}(\mathcal{O})$ and $\mathbf{N}_{\mathcal{H}(\mathcal{O})}+\mathbf{T}$ where no confusion can occur. Now a Lindenbaum lemma.

Lemma 4.3 *(Lindenbaum lemma) Let $\overline{\mathcal{H}(\mathcal{O})}$ be the hybrid logic obtained by extending the set of nominals in $\mathcal{H}(\mathcal{O})$ with a countably infinite set of new nominals and a countably infinite set of new first-order variables. Let ϕ_1, ϕ_2, ϕ_3, ... be an enumeration of all satisfaction statements in $\overline{\mathcal{H}(\mathcal{O})}$. For every $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$-consistent set of satisfaction statements Γ, a maximal $\mathbf{N}_{\overline{\mathcal{H}(\mathcal{O})}} + \mathbf{T}$-consistent set of satisfaction statements $\Gamma^* \supseteq \Gamma$ is defined as follows. Firstly, Γ^0 is defined to be Γ. Secondly, Γ^{n+1} is defined by induction. If $\Gamma^n \cup \{\phi_{n+1}\}$ is $\mathbf{N}_{\overline{\mathcal{H}(\mathcal{O})}} + \mathbf{T}$-inconsistent, then Γ^{n+1} is defined to be Γ^n. Otherwise Γ^{n+1} is defined to be*

(i) $\Gamma^n \cup \{\phi_{n+1}, a : \psi[z/x], a : E(z)\}$ *if* ϕ_{n+1} *is of the form* $a : \exists x\psi$*;*

(ii) $\Gamma^n \cup \{\phi_{n+1}, b : \psi, a : \Diamond_i b\}$ *if* ϕ_{n+1} *is of the form* $a : \Diamond_i\psi$*;*

(iii) $\Gamma^n \cup \{\phi_{n+1}, b : \psi[b/c], a : b\}$ *if* ϕ_{n+1} *is of the form* $a :\downarrow c\psi$*;*

(iv) $\Gamma^n \cup \{\phi_{n+1}, a : \psi[b/c]\}$ *if* ϕ_{n+1} *is of the form* $a : \exists c\psi$*;*

(v) $\Gamma^n \cup \{\phi_{n+1}, e : \vee_{j=1}^m (s_{j1} \wedge \ldots \wedge s_{jn_j})[\overline{d}, \overline{b}/\overline{a}, \overline{c}][\overline{t}, \overline{z}/\overline{x}, \overline{y}]\}$ *if there exists a formula in* $\mathbf{T}$ *of the form* $\forall\overline{a}\,\overline{x}((S_1 \wedge \ldots \wedge S_n) \rightarrow \exists\overline{c}\,\overline{y} \vee_{j=1}^m (S_{j1} \wedge \ldots \wedge S_{jn_j}))$ *such that* $m \geq 1$ *and* $\phi_{n+1} = e : (s_1 \wedge \ldots \wedge s_n)[\overline{d}/\overline{a}][\overline{t}/\overline{x}]$ *for some terms* $\overline{t}$ *and some nominals* $\overline{d}$ *and* e*; and*

(vi) $\Gamma^n \cup \{\phi_{n+1}\}$ *if none of the clauses above apply.*

In clause 1, z is a new first-order variable that does not occur in Γ^n or ϕ_{n+1}, in clause 2, 3, and 4, b is a new nominal that does not occur in Γ^n or ϕ_{n+1}, in clause 5, $\overline{b}$ is a list of new nominals of the same lenght as $\overline{c}$ such that none of the nominals in $\overline{b}$ occur in Γ^n or ϕ_{n+1}, and similarly, $\overline{z}$ is a list of new first-order variables of the same lenght as $\overline{y}$ such that none of the variables in $\overline{z}$ occur in Γ^n or ϕ_{n+1}. Finally, Γ^* is defined to be $\cup_{n\geq 0}\Gamma^n$.

Proof. First, Γ^0 is $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$-consistent by definition and hence also $\mathbf{N}_{\overline{\mathcal{H}(\mathcal{O})}} + \mathbf{T}$-consistent. Secondly, to check that the consistency of Γ^n implies the consistency of Γ^{n+1}, we check the first five clauses in the definition of Γ^{n+1}. This implies the consistency of Γ^*. It is straightforward to check that Γ^* furthermore is maximal consistent. □

Below we shall define a canonical model. First a small lemma.

Lemma 4.4 *Let Δ be a maximal consistent set of satisfaction statements. Let $\sim_\Delta$ be the binary relation on the set of nominals defined by the convention that $a \sim_\Delta a'$ if and only if $a : a' \in \Delta$ and let $\sim_\Delta$ be the binary relation on the set of terms defined by the convention that $t \sim_\Delta t'$ if and only if for some nominal b, $b : (t = t') \in \Delta$ (note that the notation $\sim_\Delta$ is overloaded). Then the defined relations are equivalence relations with the following properties.*

(i) *If* $a \sim_\Delta a'$ *and* $a : E(t) \in \Delta$*, then* $a' : E(t) \in \Delta$*.*

(ii) *If* $a \sim_\Delta a'$*,* $c \sim_\Delta c'$*, and* $a : \Diamond_i c \in \Delta$*, then* $a' : \Diamond_i c' \in \Delta$*.*

(iii) *If* $a \sim_\Delta a'$*,* $t_1 \sim_\Delta t'_1$*, ...,* $t_n \sim_\Delta t'_n$*, and* $a : P(t_1, \ldots, t_n) \in \Delta$*, then* $a' : P(t'_1, \ldots, t'_n) \in \Delta$*.*

(iv) *If* $a \sim_\Delta a'$, *then for any non-rigid designator* q, $(a : q) \sim_\Delta (a' : q)$.

Proof. A straightforward extension of the proof for propositional hybrid logic given in [6]. □

Given a nominal a, we let $[a]$ denote the equivalence class of a with respect to $\sim_\Delta$ (and analogously if the nominal a is replaced by a first-order variable x). We now define a canonical model.

Definition 4.5 (Canonical model) Let Δ be a maximal $\mathbf{N}_{\overline{\mathcal{H}(\mathcal{O})}}+\mathbf{T}$-consistent set of satisfaction statements. A model $\mathcal{M}^\Delta = (W^\Delta, R_1^\Delta, \ldots, R_m^\Delta, D^\Delta, \delta^\Delta, V^\Delta)$ and an assignment g^Δ for $\mathcal{M}^\Delta$ is defined as follows.

- $W^\Delta = \{[a] \mid a \text{ is a nominal of } \overline{\mathcal{H}(\mathcal{O})}\}$.
- $D^\Delta = \{[t] \mid t \text{ is a term of } \overline{\mathcal{H}(\mathcal{O})}\}$.
- $\delta^\Delta([a]) = \{[t] \mid a : E(t) \in \Delta\}$.
- $[a]R_i^\Delta[c]$ if and only if $a : \Diamond_i c \in \Delta$.
- $V^\Delta([a], q) = [a : q]$.
- $V^\Delta([a], P) = \{([t_1], \ldots, [t_n]) \mid a : P(t_1, \ldots, t_n) \in \Delta\}$.
- $g^\Delta(a) = [a]$.
- $g^\Delta(x) = [x]$.

Note that properties of the relations mentioned in Lemma 4.4 imply that the model $\mathcal{M}^\Delta$ is well-defined. Now a truth lemma.

Lemma 4.6 *(Truth lemma) Let* Γ *be a* $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$*-consistent set of satisfaction statements. Then for any satisfaction statement* $a : \phi$, $a : \phi \in \Gamma^*$ *if and only if* $\mathcal{M}^{\Gamma^*}, [a] \models \phi[g^{\Gamma^*}]$.

Proof. Induction on the degree of ϕ. □

We now just need one lemma before we can prove completeness. Recall that we are working with a fixed basic geometric theory $\mathbf{T}$. A model $\mathcal{M}$ is called a $\mathbf{T}$-*model* if and only if $\mathcal{M} \models \theta$ for every formula θ in $\mathbf{T}$ (note that the model $\mathcal{M}$ in this definition is considered a first-order model).

Lemma 4.7 *Let* Γ *be a* $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$*-consistent set of satisfaction statements. Then the model* $\mathcal{M}^{\Gamma^*}$ *is a* $\mathbf{T}$*-model.*

Proof. The proof makes use of the truth lemma, Lemma 4.6. □

Theorem 4.8 *(Soundness and completeness) The two statements below are equivalent.*

(i) ϕ *is derivable from* Γ *in* $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$.

(ii) *For any* $\mathbf{T}$*-model* $\mathcal{M}$ *and any assignment* g*, if, for any formula* $\psi \in \Gamma$, $\mathcal{M}, g \models \psi$*, then* $\mathcal{M}, g \models \phi$.

Proof. Soundness is by induction on the structure of the derivation of ϕ. Completeness is analogous to the completeness proof given in [6]. □

5 Normalisation

In this section we give reduction rules for the natural deduction system $\mathbf{N}_{\mathcal{H}(\mathcal{O})}$ + $\mathbf{T}$ and we give a normalisation theorem. First some conventions. If a premise of a rule has one of the forms $a : \Diamond_i c$, $a : E(t)$, $a : c$, or $b : (t = u)$, then it is called a *relational premise*, and similarly, if the conclusion of a rule has one of these forms, then it is called a *relational conclusion*. Moreover, if an assumption discharged by a rule has one of the mentioned forms, then it is called a *relationally discharged assumption*. The premise of the form $a : \phi$ in the rule $(\rightarrow E)$ is called the *minor premise*. A premise of an elimination rule that is neither minor nor relational is called *major*. Note that the notion of a relational premise is defined in terms of rules; not rule-instances. A similar remark applies to the other notions above. Thus, a formula occurrence in a derivation might be of the form $a : E(t)$ and also be the major premise of an instance of the rule $(\rightarrow E)$.

A *maximum formula* in a derivation is a formula occurrence that is both the conclusion of an introduction rule and the major premise of an elimination rule. Maximum formulas can be removed by applying *proper reductions*. The rules for proper reductions are the rules for proper reductions for propositional hybrid logic given in [6] together with the rule below that deals with the case where $(\forall I1)$ is followed by $(\forall E1)$.

$$\dfrac{\dfrac{\begin{array}{c}[a : E(z)]\\ \vdots\ \pi_1 \\ a : \phi[z/x]\end{array}}{a : \forall x \phi} \qquad \begin{array}{c}\vdots\ \pi_2\\ a : E(t)\end{array}}{a : \phi[t/x]} \quad \rightsquigarrow \quad \begin{array}{c}\vdots\ \pi_2 \\ a : E(t) \\ \vdots\ \pi_1[t/z] \\ a : \phi[t/x]\end{array}$$

We also need reduction rules in connection with the (R_θ) inference rules. A *permutative formula* in a derivation is a formula occurrence that is both the conclusion of a (R_θ) rule and the major premise of an elimination rule. Permutative formulas in a derivation can be removed by applying *permutative reductions*. The rules for permutative reductions are similar to the rules for permutative reductions for propositional hybrid logic given in [6].

A derivation is called *normal* if it contains no maximum or permutative formula.

Theorem 5.1 *(Normalisation) Any derivation in* $\mathbf{N}_{\mathcal{H}(\mathcal{O})} + \mathbf{T}$ *can be rewritten to a normal derivation by repeated applications of proper and permutative reductions.*

Proof. The first step of the theorem is to prove that any derivation can be rewritten to a derivation in which all maximum or permutative formulas are of the form $a : \Diamond c$ or $a : E(t)$. This is done using a variation of a standard technique (originally given in [10]).

The second step of the theorem is to prove that any derivation which is the result of the first step can be rewritten to a derivation in which all maximum or permutative formulas are of the form $a : \neg c$ or $a : \neg(t = u)$. This is done using a technique introduced in [6] (where it is used to solve a similar problem in connection with propositional hybrid logic).

The third step of the theorem is to prove that any derivation which is the result of the second step can be rewritten to a normal derivation; this is done using the standard technique mentioned in the first step of the theorem. □

It can be proved that any normal derivation satisfies a version of the subformula property; see [6] for the propositional case.

6 Comparison to related work

Our natural deduction systems share several features with the tableau system for first-order hybrid logic given in [4], for example the feature that formulas in derivations in general are satisfaction statements, that is, formulas of the form $a : \phi$. Since our systems are in natural deduction style, we provide a proof-theoretic analysis in the form a normalisation theorem.

The use of geometric theories in the context of proof-theory traces back to [13] where it was pointed out that basic geometric theories correspond to simple natural deduction rules for intuitionistic modal logic. A natural deduction system for classical modal logic which is similar to the system of [13] has been given in [1]. The latter paper only considers Horn clauses, however. In [2], the system of [1] is generalised to the first-order case. In the natural deduction system considered in [2], a distinction is made between the language of ordinary first-order modal logic and a metalanguage involving atomic formulas of the form $R(a, c)$ (meaning the world a is R-related to the world c); atomic formulas of the form $a : t$ where t is a term of ordinary first-order modal logic (meaning the individual t exists at the world a); and formulas of the form $a : \phi$ where ϕ is a formula of ordinary first-order modal logic (meaning the formula ϕ is true at the world a). One contribution of the present paper is to point out that basic geometric theories correspond to natural deduction rules for first-order hybrid logic where no such distinction between an object language and a metalanguage is made. It is notable that the normalisation theorem given in the present paper is more complicated to prove than the one given in [2]; the reason being that the above mentioned formulas $R(a, c)$ and $a : t$ in the system of [2] cannot be maximum formula as they are atomic whereas their hybrid logical counterparts $a : \Diamond c$ and $a : E(t)$ can be maximum formulas in the systems considered in the this paper, and moreover, such maximum formulas cannot be removed using the standard technique but requires another technique (see the proof of the normalisation theorem, Theorem 5.1).

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